Geometric Constructions with the Compass.

A Thesis submitted to the faculty of the Graduate School of the University of Minnesota

by

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1 9 1 1.

Index of Figures.

Abreviations used:

M --Mascheroni's construction.

P. -- Construction from equations or by plotting

I. -- Compass construction based on theory of inversion

XXl. I. Inverse of a line

XXll. I. Inverse of a circle.

XXlll. I. Divide a line into two equal parts.

XXlV. I. Divide a line into three equal parts.

XXV. I. Intersection of two lines

XXVl. I. Intersection of a line and circle.

XXVll. I. Find centre of a given circle.

XXVlll. I. Construct an angle equal to a given angle.

XXlX. I. Inscribe a square in a circle.

XXX. l. Find mean proportional.

XXXl. I. Inscribe three equal circles in a given circle.

XXXll. M. Divide a line into two equal parts.

XXXlll. M. Divide a line into three equal parts.

XXXlV. I. Erect a perpendicular to a line at its extremity.

XXXV. I. Find mean proportional.

XXXVl. M. Divide a line internally into mean and extreme ratio.

Introduction.

In the last years of the eighteenth century appeared a book by an Italian mathematician, Mascheroni, called "Geometria del Compasso," or "Geometry of the Compass." It is a body of constructive geometry in which the use of the circle alone is postulated. Many of the solutions are very ingenious, and some of the constructions of considerable practical importance. The aim of his treatise was to exhaust the subject and give all the elements for all possible cases and show that with compass alone could be found all the points that could be found with the aid of ruler and compass He did not attempt to solve all problems of elementary geometry but after having shown the elements necessary and sufficient for all, he gave a great number of the principal ones, especially those which seemed most useful or preferable on account of their elegance.

Lorenzo Mascheroni was born at Bergamo, Italy in 1750. He had a great liking for literature and attained such brilliant success in his first studies that at the age of eighteen he was made teacher of Greek and Latin at the College of Bergamo. He was soon called to the University of Pavia, founded by Charlemagne, to fill the chair as professor of Greek. He remained there until he was twenty-seven years old, but like Lagrange it was not in literature that he was to become famous. One day while at his work at the University of Pavia he came across a work on mathematics which he read and reread with great pleasure. From that day he had such a love for science that it became almost a passion with him, and he made the exact sciences the principal object of his studies. His progress was rapid. In a short time he was appointed to the chair of geometry at the College of Bergamo, and a little later he was named professor of geometry and algebra at the University of Pavia.

In 1787 he published a treatise called "The method of Measuring Plane Polygons", and six years later a work composed of 5 books under the modest title "Problems for surveyors with different solutions." The latter was hardly more than a new edition of the former publication with an additional book on the measurement of solids. In the preface of this work Mascheroni had a chance to reclaim authorship of his first work which seems to have been copied by another mathematician and published in 1789 at Geneva under the title "Polygonometrie de Lhuiller".

On the entrance of the French into Italy Macheroni, on account of his talents and wisdom, was made member of the legislative body of the Cisalpine Republic by its citizens, and in this position he came in close touch with Napoleon Bonaparte during his stay in Italy. Nevertheless Mascheroni still continued his mathematical studies and in 1797 published at Pavia the "Geometry of the Compass", his best known mathematical work. It attracted the attention of the General who was often seen talking with the learned Italian about his new book.

In December of the same year Bonaparte, at a meeting of the members of the Institute of France, brought to the notice of Lagrange and Laplace Mascheroni's new work on the Geometry of the Compass, which was as yet unknown in France. Not long after, Mascheroni's work was translated into French by one of the members of the Institute.

In 1798, while professor of mathematics at the University of Pavia, Mascheroni was sent to Paris by the Italian Government to cooperate in the drawing up of a new system of weights and measures. The progress of the war being such as not to permit him to return to his country, he tried to exercise his talents in France. But the fatigue of the works to

~~to~~ which Mascheroni had devoted his whole life had ruined his health and in 1800 in spite of all that doctors could do he passed away at the home of M. Dubois who gave him the best of care during his sickness.

Delambre, the celebrated French geometer, did the Italian geometer justice in the historical report which he made in 1808 on the progress of mathematical science from 1789 to 1808. In the introduction to the report he recalled-that ancient geometry admitted in its demonstrations only that which could be done by ruler and compass. The following is a translation from his report which was made in the name of the Instutute of France.

"Mascheroni, still more rigid, wished to do without the ruler. Anyone is surprised at the great number of new and useful propositions which he has been able to find in a subject seemingly so exhausted. His principal thorems were brought into France by the Conqueror and pacifier of Italy. Everyone was anxious to be acquainted with the whole work and a French translation of it soon appeared".

Just as it was Mascheroni's aim in his Geometry of the Compass, so I, in this thesis purpose to show that there is no problem of elementary geometry which can not be solved by compass alone, in the sense that the compass is sufficient to find all the points necessary to determine the position and size of all straight lines necessary for the solution of the given problem. Unlike Mascheroni, however, I base all my constructions on a certain theory, called the theory of inversion(later discussed), while he, in a great many instances, relied on chance for his solution.

Account of Mascheroni's Work.

Mascheroni in his Geometry of the Compass says "I call Geometry of the Compass that which by means of the Compass alone and without the aid of the ruler determines the position of points. "To the geometry of the Compass belong all the problems that one can solve with the Compass alone although one may not be able to prove them by construction lines drawn with that instrument alone. He had two aims in writing his treatise, first to show that there was no problem of elementary geometry which could not be solved by compass alone, and second to devise a method using only the compass, for dividing the circumference of a circle into any number of equal parts for the aid of astronomers. His work was based on no definite plan or theory; he acknowledged that he owed much to chance and that often it was only after many different attempts that he obtained the result sought for.

The ancients after dividing the circumference into six equal parts by compass alone obtained other points of division with ruler and compass by taking different points outside the circumference. Mascheroni, by an easier and more exact method than that of the Ancients, determined by Compass alone three points a,b,e, not on the circumference, which might be called remarkable points for any means of them alone and only five different radii he divided the circumference into two hundred and forty equal parts, or into any number of equal parts which is a factor of 240 such as 2,3,4,5,6,8,10 etc. The point a, which alone suffices to divide the circumference into four, eight, twelve or twenty four equal parts, using as radii $\sqrt{1}$=AB, $\sqrt{2}$,=Aa, $\sqrt{3}$=B D, determined from points in the figure, is found as follows: (See figure 1) In the circle A(A B)(A denotes center and A B radius) make A B=Bc=BC=CD=DE=Ed. Also make B D=B a=E a.
(It is to be understood that this abreviation means with a radius B D and

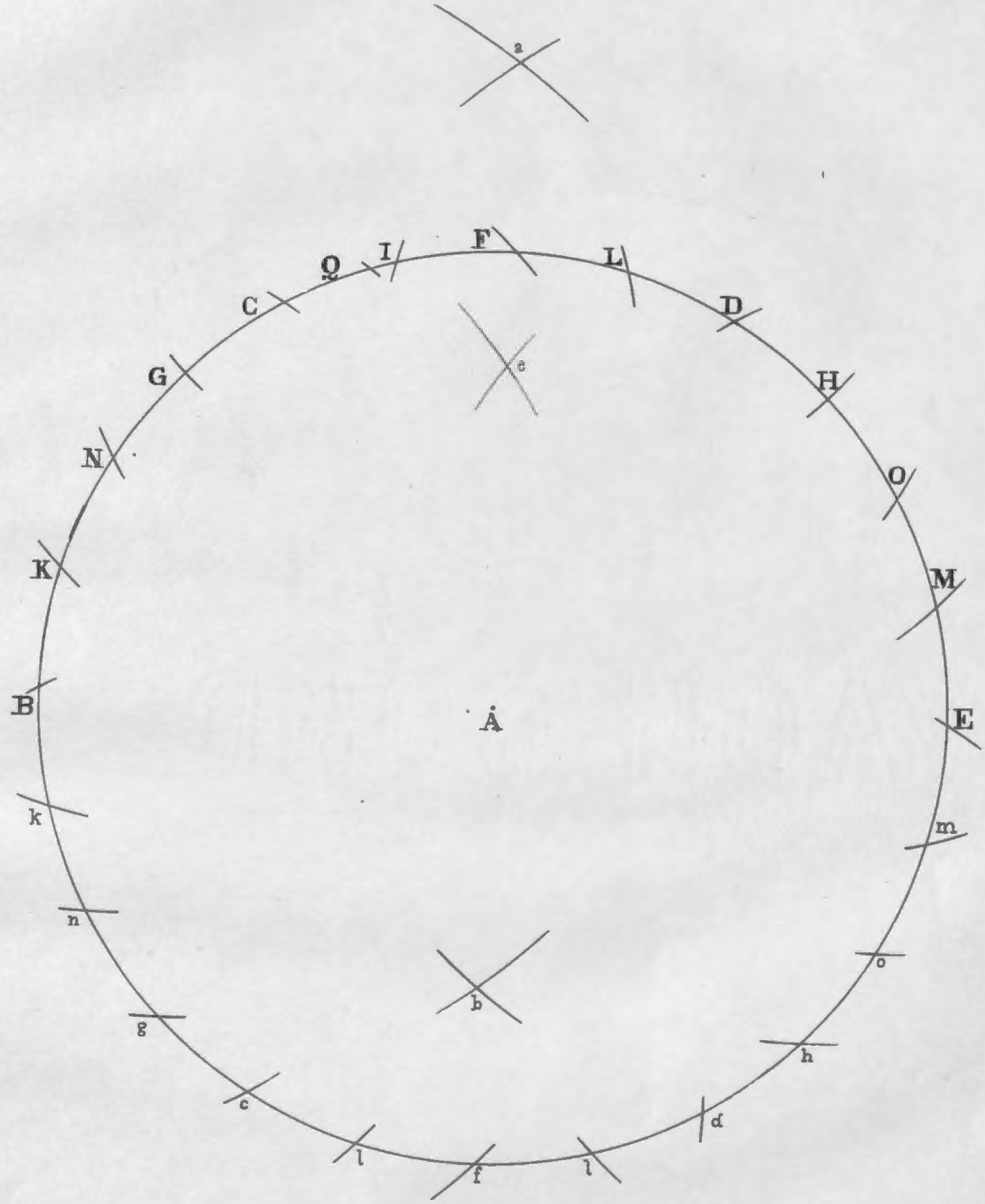


FIGURE I

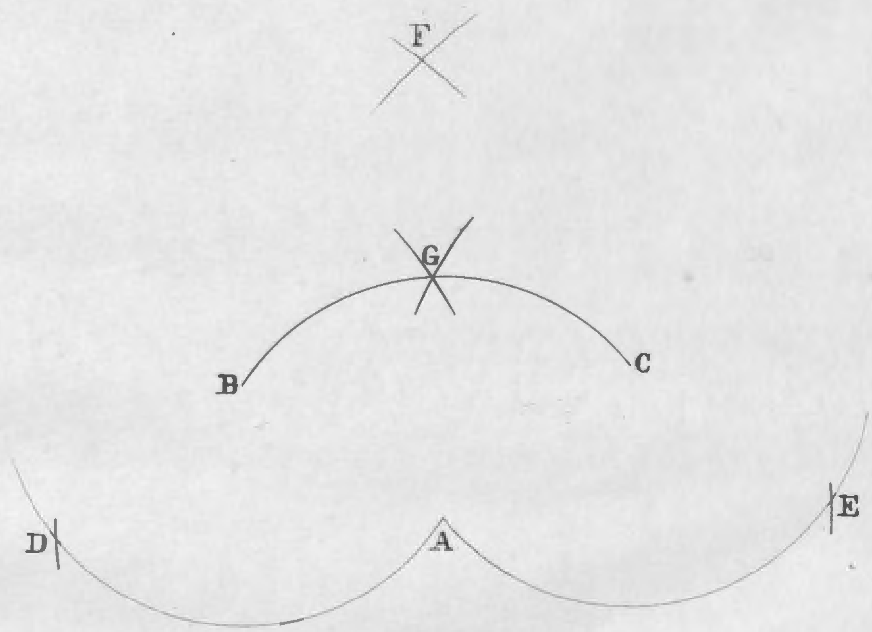


FIGURE II

center B describe an arc containing the point a ; with the same radius B D and center E describe another arc cutting the first at the point a.)

If Aa=BF=Bf then the circumference will be divided into four equal parts at the points B,F,E and f. If AB=aG=aH and Aa=Gg=Hh the quadrants already obtained will be bisected at G,H,h,g. If AB=FN=Nn=FO=Oo the six equal arcs B C, C D etc. will be bisected at N,F,O,o,f,n. If AB=GL=LM=G k=k i=H I=IK=H m=m l the problem of dividing the circumference into 24 equal parts will have been solved.

I shall now show how the points b and e, for finding other points of division of the circle, are obtained without showing how they aid in dividing the circumference into 10,20,48,120 or 240 equal parts.

Making use of the points A,a,N,O already found make Aa=N b=Ob and a N=B e=E e. However it may be interesting to show how easily the circumference can be divided into five equal parts. Using the same notation as before the arc B Q=B b is one fifth of the circumference.

An important preliminary problem which has to do with division of arcs is that of bisecting any arc. Problem: To divide any arc B C of circle A (A B) into 2 equal parts (see figure 2) .

Solution:- With centres B and C and radius A B describe arcs A D and A E. Make B C=A D=A E. Then with centres D and E and radius D C=B E describe arcs which cut in F. With same centres D and E and radius A F describe two arcs which intersect at G. G is the required point.

If the arc to be bisected is greater than a semicircumference it is necessary to subtract equal arcs from each end and bisect the remaining arc. The demonstration of this problem is not hard but is too long to be given in this summary.

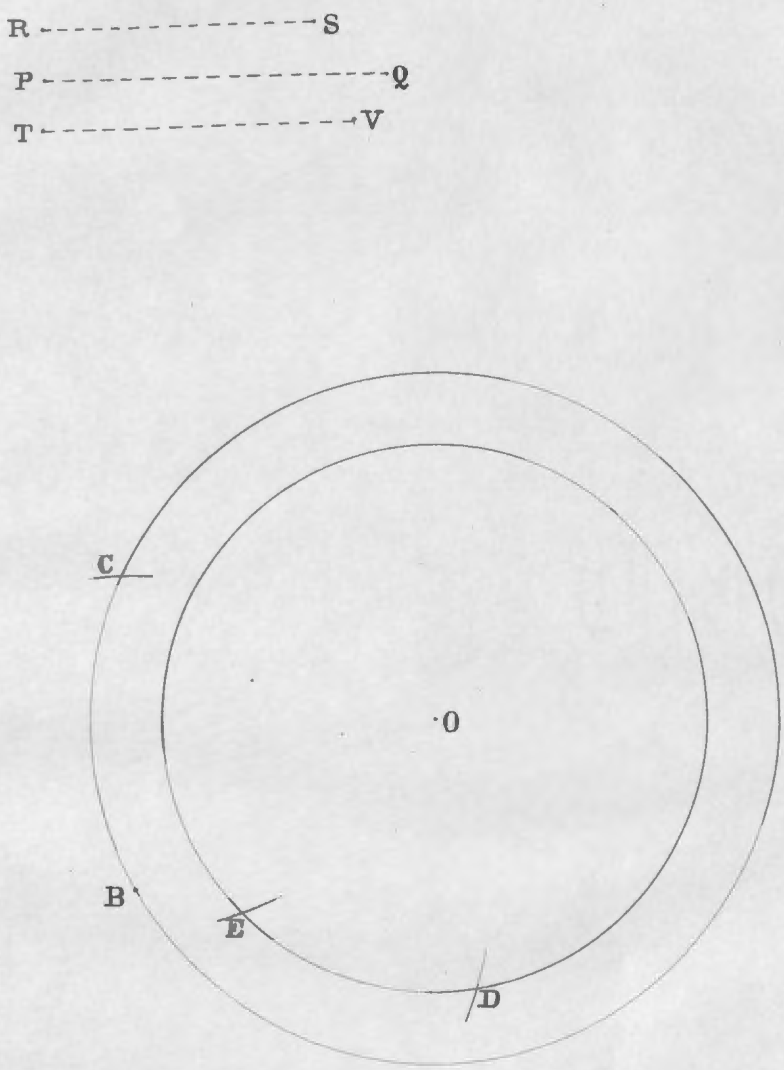


FIGURE III

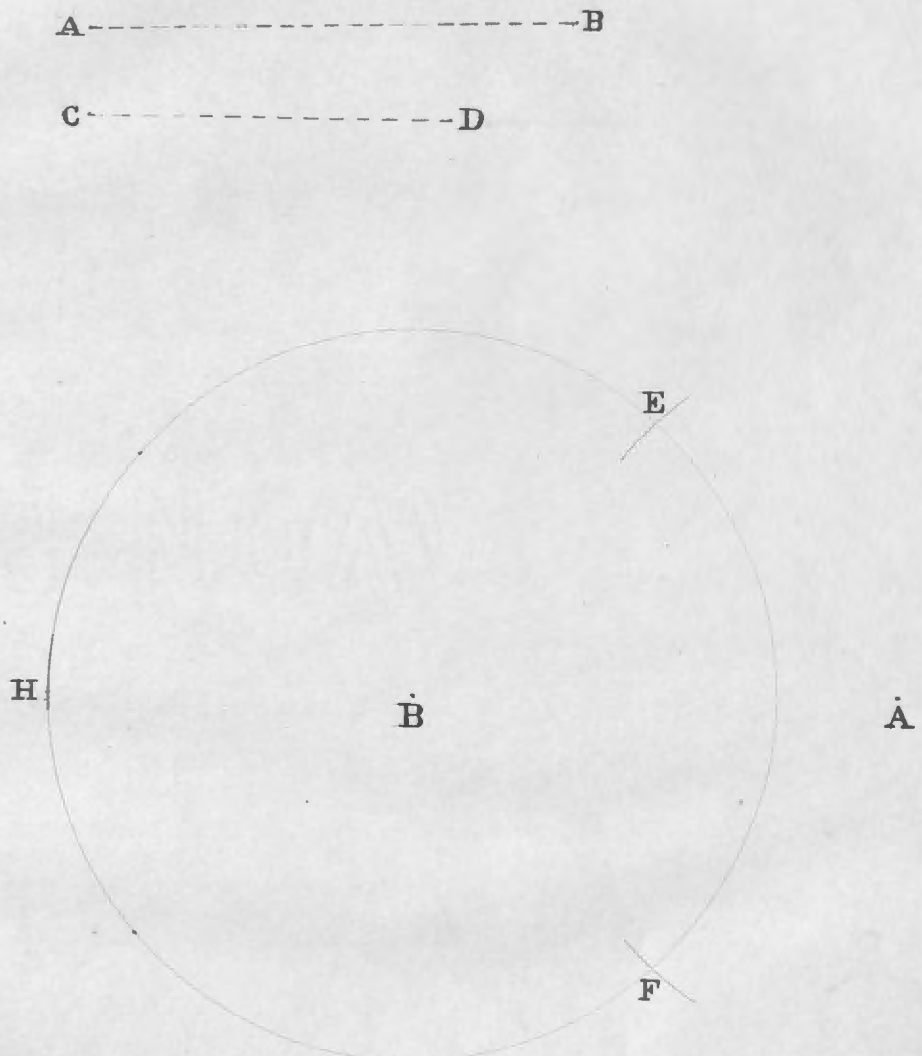


FIGURE IV

By a method very similar to that based on the theory of inversion Mascheroni shows how to multiply or divide any finite line(this will be discussed later in Part 3.) and also how to find third, fourth or mean proportionals so as to divide any straight line into parts having any given ratio.

Problem: -To find the fourth proportional to three distances P Q,R S, T V. (See figure 3.)

Solution:- With any centre O and P Q and R S as radii describe two concentric circles B C and D E. With T V as a radius and any point B of the first circle as centre describe an arc which cuts it at C. With the same centre and any arbitrary radius describe an arc which cuts the second circumference at D. With the same radius B D and centre C cut the same circumference at E. Then D E will be the required fourth proportional.

The demonstration of this problem depends on the similarity of triangles but will not be given here. Book lV teaches how to find the points for all cases of perpendiculars and parallels and how to add or subtract finite distances from other finite distances, not because it is easier with compass alone but to show that there is no problem of elementary geometry that can not be solved with compass alone in the sense already explained.

I shall give here the constructions for the three most important problems of this book.

Problem: - To add to the line A B the distance C D. (See figure lV)

Solution:- With centre B and radius C D describe a circle; with center A and the same radius describe an arc which cuts the circle at E and F. Bisect the arc E F at H. Then H is the required point i.e. A H is the sum of A B and C D.

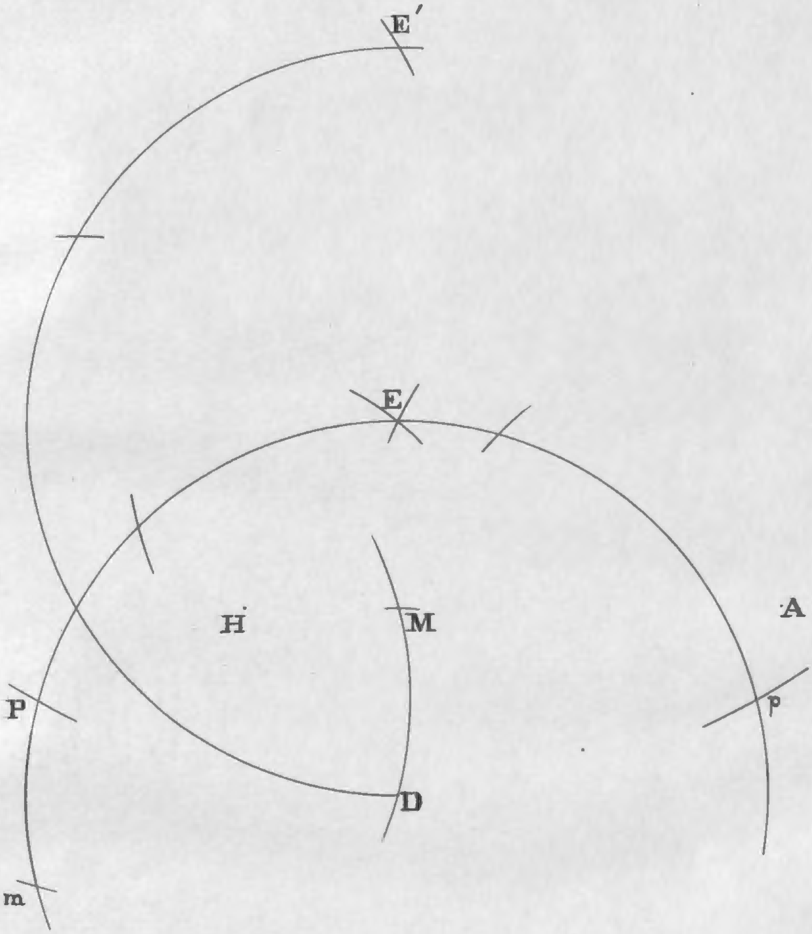


FIGURE V

FIGURE VI

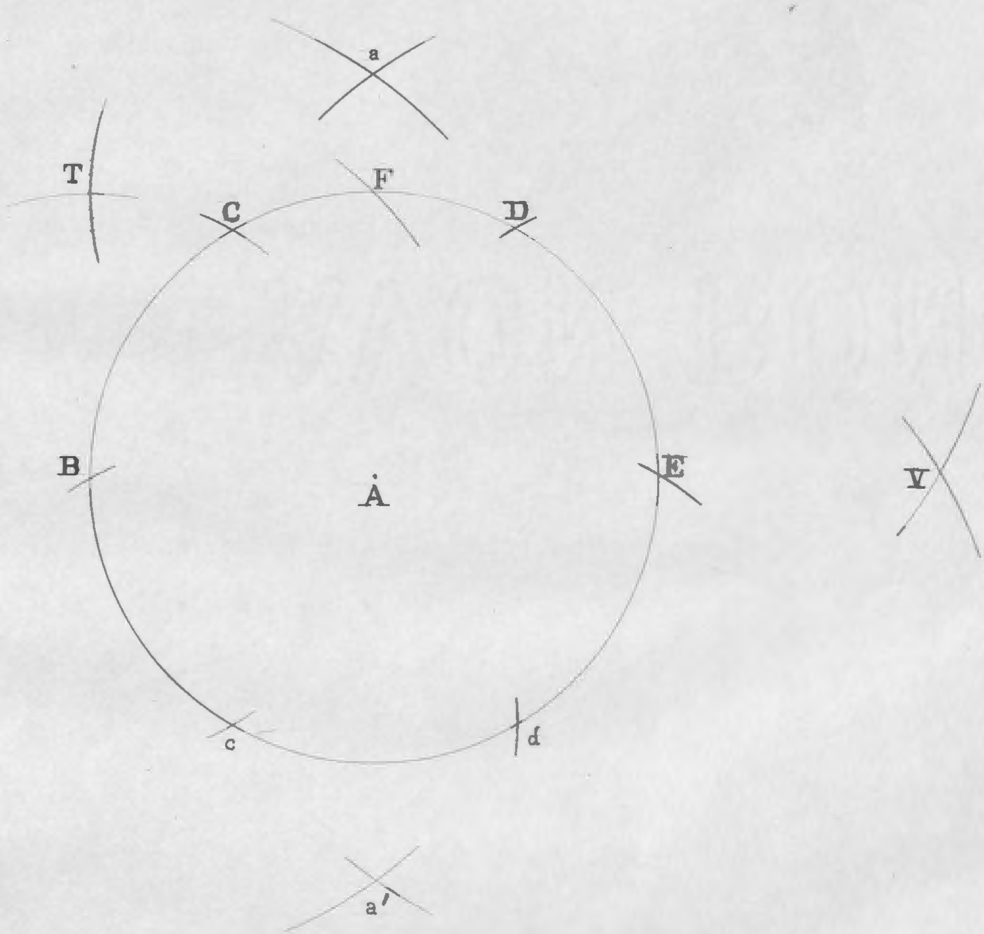


FIGURE VII

Problem:- To erect the perpendicular to A B from D and find the point M where it cuts the line A B. (See figure V.)

Solution:- Make A D=A E and B D = B E.

Point E will be the first point sought.

The middle point of D E will be the point M where the perpendicular cuts A B. (The solution for bisecting a line is given later on Page .)

Problem:- To draw through C the line parallel to A B. (See figure Vl.)

Solution:- Make C A=B D and B A=C D.

The point D will be the point sought.

Book Vl. is very interesting in that by taking the radius of the circle equal to unity the square root of any whole number can easily be found. We have already obtained the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, The method can be extended for any number but the following table together with figure Vll. is sufficient to show the plan.

The points of figure Vll. are obtained as follows:-

A B=B C=C D=D E=E d=d c

With B and E as centres and B D as radius describe arcs which intersect at a and a'. With the same radius B D and centres D and d describe arcs which intersect at V. With radius A a and centre B cut the circumference at F. With centres B and F and radius A B describe two arcs which intersect at T.

Table for square root of numbers up to ten.

A B=$\sqrt{1}$	a V=$\sqrt{6}$
A a=$\sqrt{2}$	C V=$\sqrt{7}$
B D=$\sqrt{3}$	a a'=$\sqrt{8}$
B E=$\sqrt{4}$	B V=$\sqrt{9}$
E T=$\sqrt{5}$	T V=$\sqrt{10}$

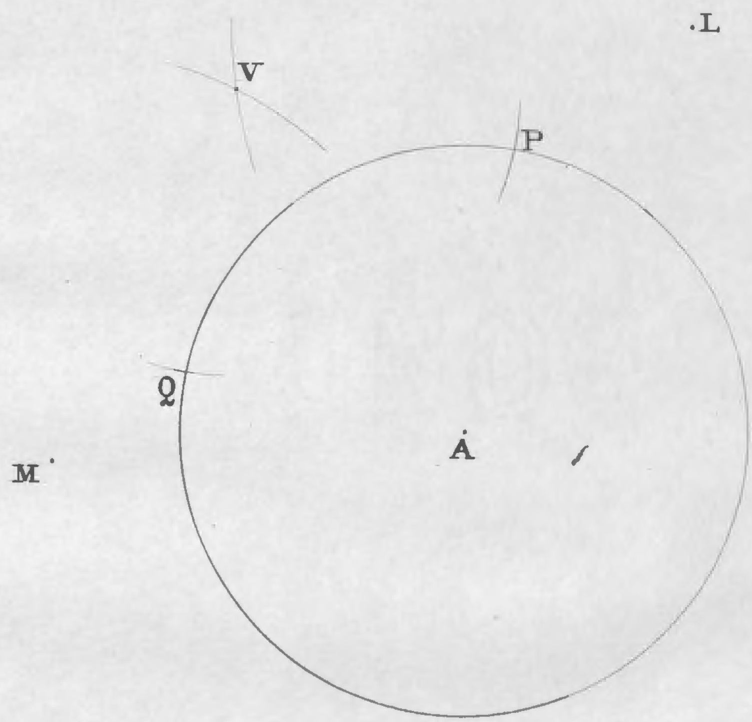


FIGURE VIII

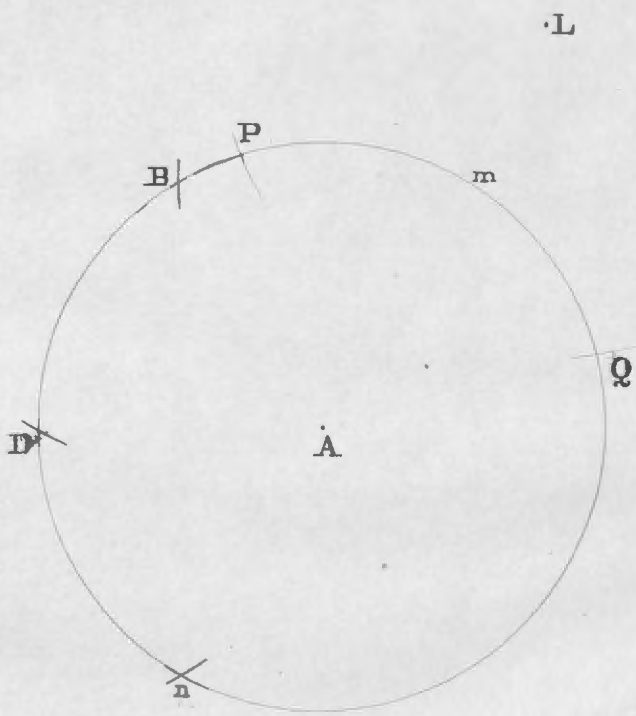


FIGURE IX

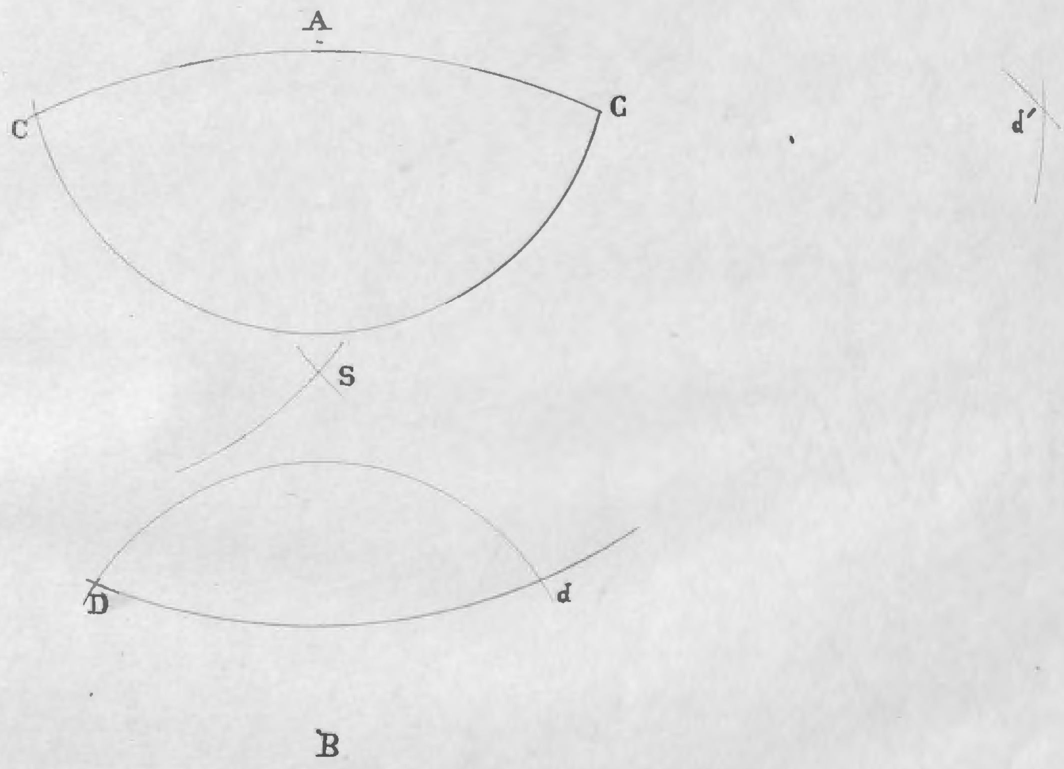


FIGURE X

The remaining books show how to find the intersections of straight lines with arcs of circles, and of straight lines each determined by two points, give all that is necessary to determine the position of straight lines at any angle and also show how to construct similar figures and regular polygons. Inasmuch as the first two problems are fundamental ones, we shall give Mascheroni's constructions without proof however. Problem. To find the points where a straight line cuts the circle A (A B).

Case 1. The straight line L M does not go through the centre of the circle. (See figure Vlll.)

Solution:- With centres L and M and radii L A, M A respectively describe arcs which intersect at V; with centre V and radius A B describe an arc which cuts the circle at P and Q. P and Q are required points.

Case 2. The straight line L A goes through the centre. (See figure lX.)

Solution:- With centre L and any radius describe an arc which cuts the circle at P and Q; bisect arc P Q at m and determine semicircumference m D n. Then m and n are the required points.

Problem:- To find the point of intersection of two straight lines A B and C D. (See figure X)

Solution:- With centers A and B and radii A C, A D and B C, B D respectively describe four arcs which intersect at c,C,d and D. Find d' of parallelogram C D d d'; then find the fourth proportional to the three distances c d',C D, and Cc. With this distance as a radius and centres c and C describe arcs which cut at S. Then point S is the required point.

Book Xl. takes up miscellaneous problems e.g. To inscribe a regular hexagon in an equilateral triangle, to describe a spiral composed of several arcs of a circle, to form a right triangle whose sides are in arithmetic or geometric progression, to inscribe seven equal regular hexagons in a circle etc.

A great part of Book XII. is given over to constructions for dividing the circumference of a circle into four hundred degrees and subdivisions of degrees in order that the quarter of a circle which is the foundation of trigonometry may be divided into one hundred parts or degrees, each degree to be divided into 100 minutes etc.

Mascheroni suggests that although all the elementary problems solved by compass alone have not a very simple solution, yet the greater part of the more necessary ones are solved with enough brevity and simplicity to induce practical men to reject the aid of the ruler and use the compass alone to find the foundational points and then draw if necessary the straight lines from one point to another which cannot be traced with compass alone but which demand the aid of the ruler.

His proof for the statement that all ruler and compass constructions <u>can</u> be made by compass alone is embodied in the following. Any straight line necessary to the solution of a problem is determined in size and position by a combination of any two of the following: the intersection of two arcs, of an arc and a straight line or of two straight lines. The first is essentially geometry of the compass and compass solutions of the other two have been given on page 21. These two fundamental constructions together with some essential preliminary problems, most of which have been mentioned in the summary(e.g. to bisect an arc, to add one distance to another, to erect a perpendicular to a given line etc.,) are in the nature of the case sufficient to solve any problem of elementary geometry which can be solved by ruler and compass.

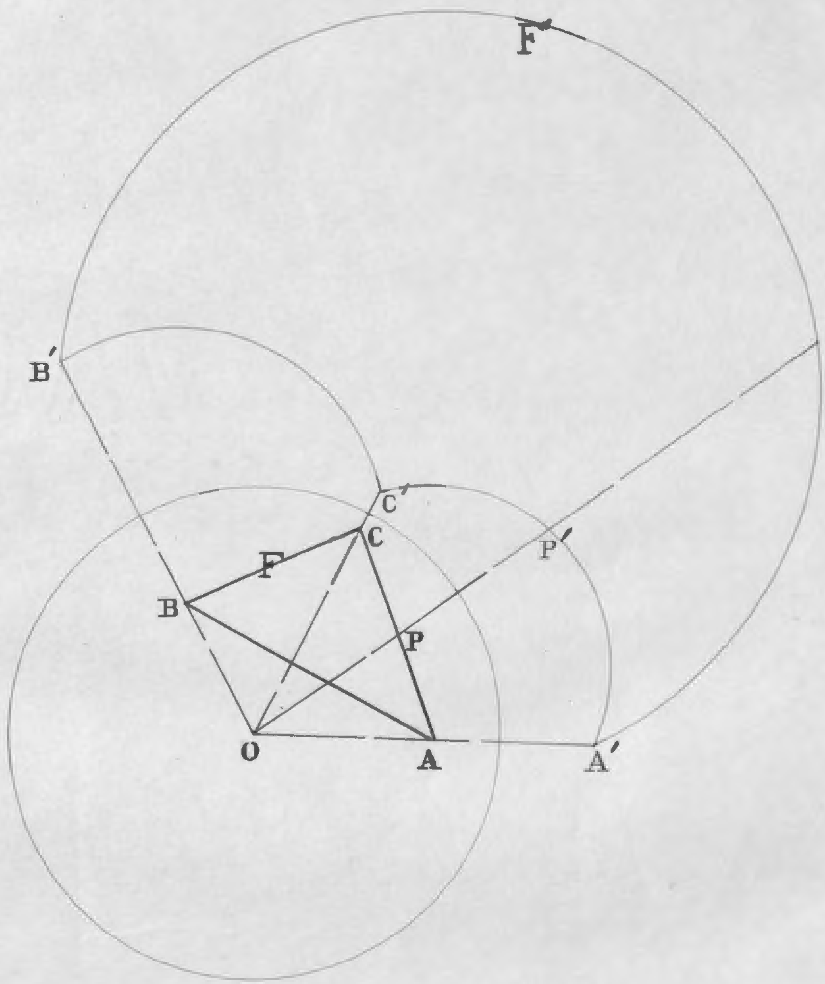


FIGURE XI

Theory of Inversion.

In order that the principles upon which the later problems are based, may be more easily understood, a short account of the theory of inversion is given here together with a few illustrations of inverse curves.

Any operation which replaces a given figure by a second figure in accordance with a given law is called a transformation. If a transformation replaces the points of one figure by the points of a second it is called a point transformation. If a point transformation replace P(x,y) by P' (x',y') then the equations expressing x' and y' in terms of x and y or conversely are called equations of transformation. Let O be the center of a circle C with radius r, and let P be any point of a given figure F.(see figure Xl.)

Construct P' on OP such that

$$OP' \cdot OP = r^2$$

By letting P assume different positions on F, P' will move on a figure F'. The operation or point transformation which in this case replaces P by P' is called an inversion, and F and F' are called inverse figures. O is called the center of inversion and C, the radius of which will be taken equal to unity unless otherwise specified is called the circle of inversion.

As shown in figure Xl. the inverse of a triangle is a figure bounded by three curves. Hence we may expect to find that the properties of inverse figures are in general quite different from those of equal or similar figures.

Two important properties are immediately evident from the definition.

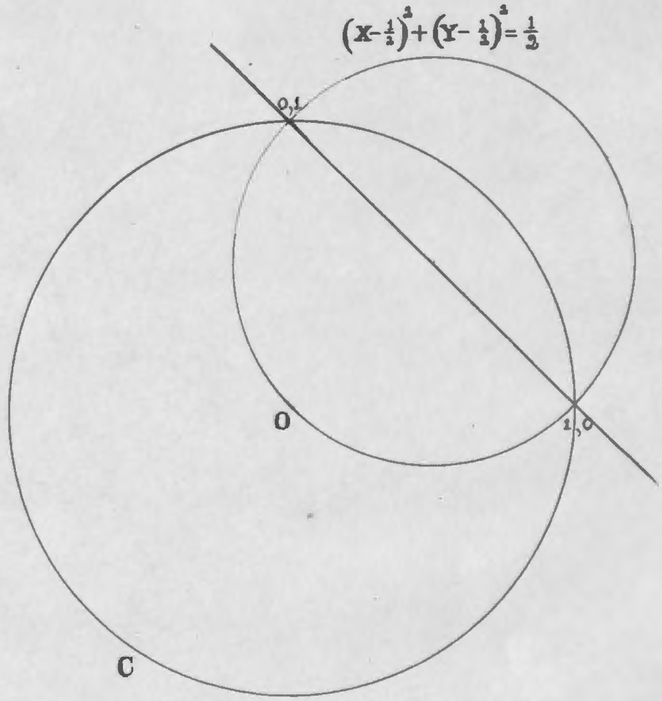


FIGURE XII

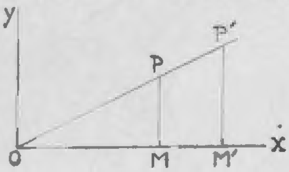
(1) If P approaches the origin, P' recedes to infinity and conversely.

(2) The points of a circle of unit radius whose center is O are fixed points. We thus see that the points in which a figure cuts this circle will be points of the inverse curve. The equations of inversion must express the two conditions that

(1) P and P' be on a line through the center.

(2) $OP' \cdot OP = r^2 = 1$.

The first of these conditions is satisfied when the triangles O P M and O P' M' are similar whence



$$(1)\quad \frac{x}{x'} = \frac{y}{y'} = \frac{OP}{OP'}$$

The second condition may be written by dividing by $(OP')^2$

$$(2)\quad \frac{OP}{OP'} = \frac{1}{(OP')^2} = \frac{1}{x'^2+y'^2}$$

From (1) and (2)

$$\frac{x}{x'} = \frac{y}{y'} = \frac{1}{x'^2+y'^2}$$

$$\therefore \quad \frac{x}{1} = \frac{x'}{x'^2+y'^2} \quad \text{and} \quad \frac{y}{1} = \frac{y'}{x'^2+y'^2}$$

Hence we have that the equations of an inversion whose center is the ~~is the~~ origin are

$$\frac{x}{1} = \frac{x'}{x'^2+y'^2} \qquad \text{and} \quad \frac{y}{1} = \frac{y'}{x'^2+y'^2}$$

Let us now work a few examples applying these equations of inversion.

(1) Find the inverse of the straight line $Ax+By+C=0$.

Solution: (See figure xll.)

Substitute in the given equation the values of x and y from the equations of inversion. This gives $\frac{Ax'}{x'^2+y'^2} + \frac{By'}{x'^2+y'^2} + \frac{C}{1} = 0$

Simplifying and dropping primes $C x^2 + C y^2 + A x + B y = 0$.

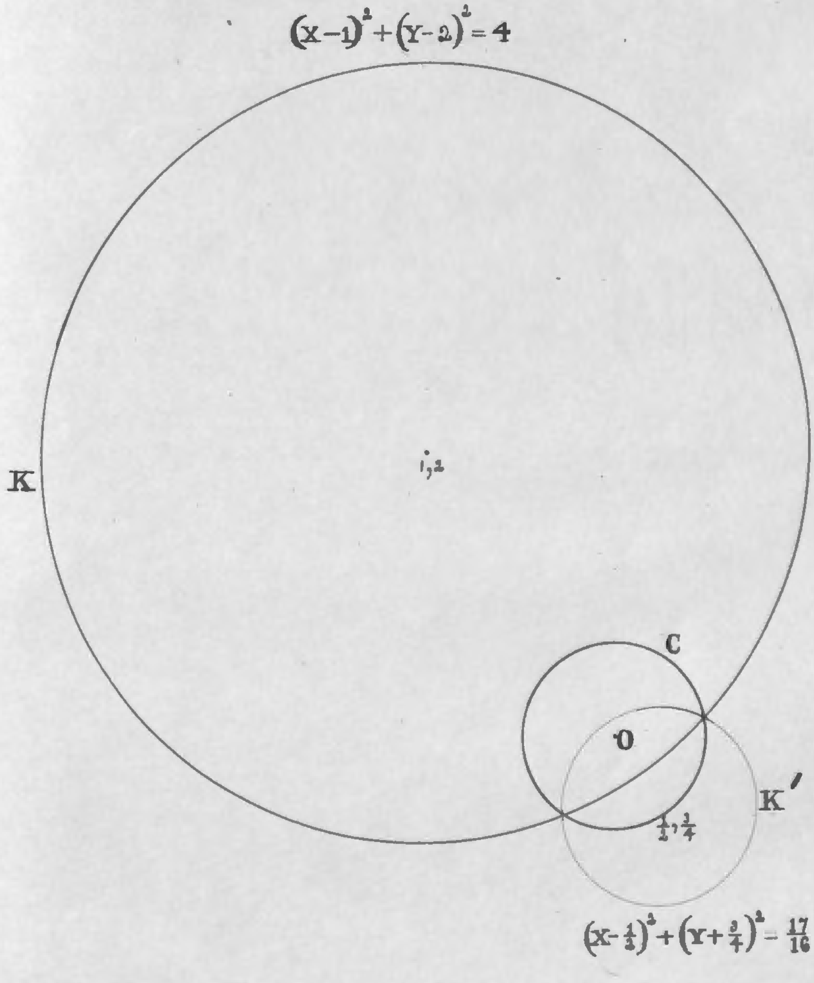


FIGURE XIII

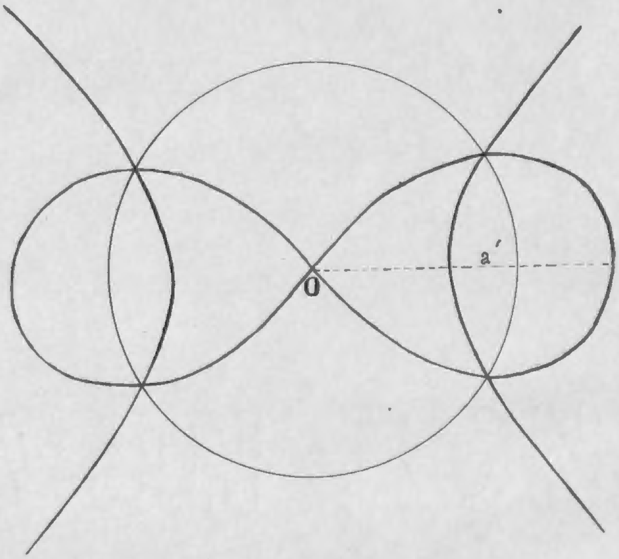


FIGURE XIV

The locus of this equation is a circle which passes through the origin If C=0 the locus is seen to be the given line. Hence the inverse of a straight line which does not pass through the origin is a circle and a line which passes through the origin is invariant under an inversion.

__ll.__ Find the inverse of the circle $x^2+y^2+Dx+Ey+F=0$.

Solution: (see figure Xlll.)

Substituting from equations of inversion we get

$$\frac{x'^2}{(x'^2+y'^2)^2}+\frac{y'^2}{(x'^2+y'^2)^2}+\frac{Dx'}{x'^2+y'^2}+\frac{Ey'}{x'^2+y'^2}+F=0.$$

Multiplying by $(x'^2+y'^2)$ and dropping primes

$$Fx^2+Fy^2+Dx+Ey+1=0.$$

The locus is a circle unless F=0 in which case it is an equation of the first degree and its locus is a straight line. Hence,

The inverse of a circle is in general a circle but the inverse of a circle which passes thru the origin is a straight line.

lll. Find the inverse of an equilateral hyperbola if center of hyperbola is center of inversion.

Solution: (See figure XlV.)

The equation of the equilateral hyperbola is

$$x^2-y^2=a^2$$

Substituting from equations of inversion we get the equation of the inverse curve is

$$\frac{x'^2}{(x'^2+y'^2)^2}+\frac{y'^2}{(x'^2+y'^2)^2}=a^2$$

Reducing and dropping primes,

$$(x^2+y^2)^2=\frac{1}{a^2}\,.\,(x^2-y^2)$$

The locus is the lemmiscate of Bernoulli.

Replacing $\frac{1}{a^2}$ by a'^2 we get the form of the equation usually given namely:

$$(x^2+y^2)^2=a'^2(x^2-y^2)$$

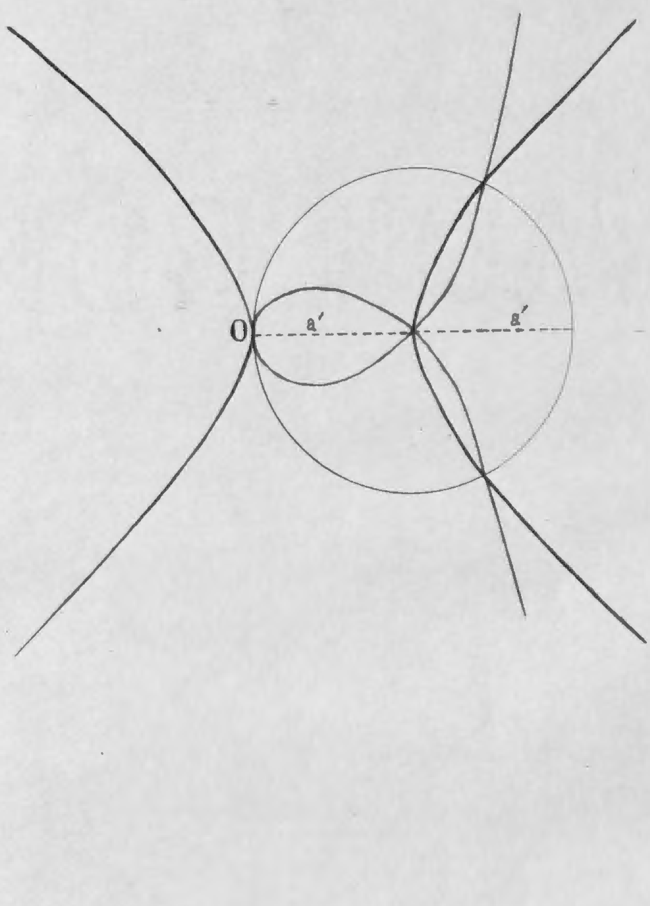


FIGURE XV

1V. Find the inverse of the equilateral hyperbola if one vertex is the center of inversion i.e. if the origin is at one vertex.

Solution: (See figure XV.)

The equation of the equilateral hyperbola when the origin is at the right hand vertex is

$$x^2 - y^2 + 2ax = 0.$$

Then the inverse curve is

$$\frac{x'^2}{(x'^2+y'^2)^2} - \frac{y'^2}{(x'^2+y'^2)^2} + \frac{2\,a\,x'}{(x'^2+y'^2)} = 0.$$

Reducing and dropping primes

$$x(x^2+y^2) + \frac{1}{2a} \cdot (x^2 - y^2) = 0.$$

The locus of this equation is the strophoid. Replacing $\frac{1}{2a}$ by a' and solving for y^2 we get the form of the equation usually given.

$$y^2 = \frac{a'+x}{a'-x} x^2$$

In general the inverse of the locus of an equation of the second degree is a curve whose equation is of the fourth degree since for every term of first degree is substituted a term of second degree. If however the given locus passes through the origin i.e. there is no absolute term the inverse equation can be reduced to third degree since the whole equation can be divided by the factor $\frac{1}{(x'^2+y'^2)}$. The equilateral hyperbola which inverts into a strophoid is an illustration of this reduction.

The only transformation required for polar coordinates is to replace r by $\frac{1}{r}$ since the angle for two corresponding radii vectores must be the same.

Let us take the polar equation of the circle which passes through the origin namely: $r = a \cdot \cos\theta$

The inverse curve would then be

$r \cos\theta = K$, the equation of a straight line.

The polar equation of a circle whose center is the origin is $r = \pm K$.

Replacing r by $\frac{1}{r}$ we get as the inverse equation.

$r = \pm\frac{1}{K}$ another circle which evidently agrees with the transformations made in rectangular coordinates.

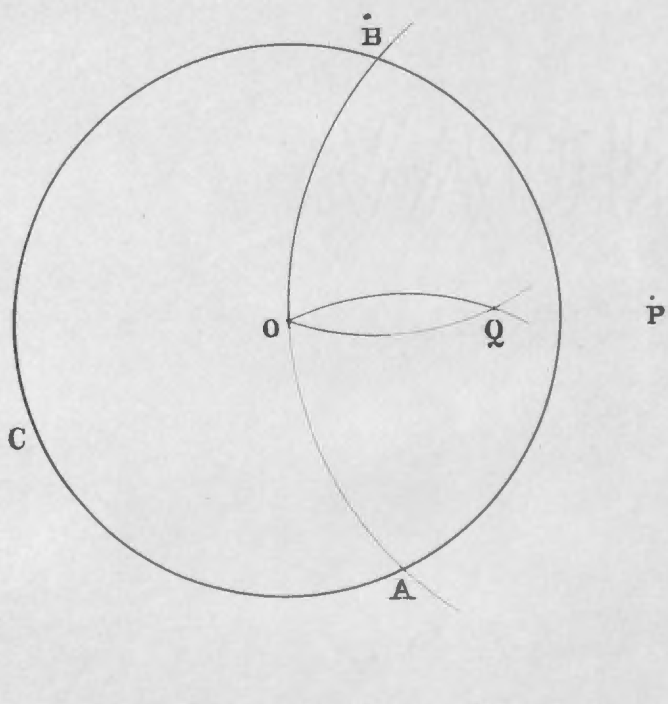


FIGURE XVI

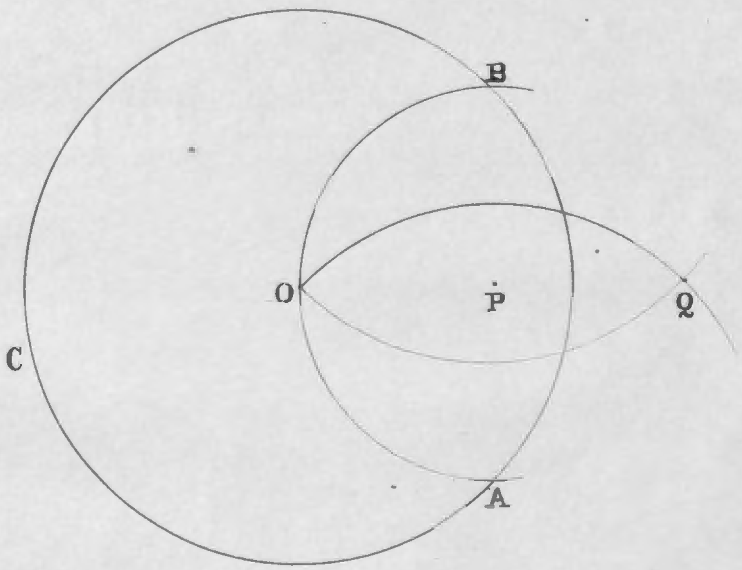


FIGURE XVII

Compass constructions by means of Inversion.

By means of the theory of inversion any ruler and compass construction of elementary géometry can be solved by the compass alone, in the sense that with the compass alone can be found all the points asked for by the problem in order to determine the needed straight lines. For example, when we say construct an angle a b c with compass alone, we mean find with the compass three points a,b,c,(or one of these points being given find the others) so that if the lines a b and b c be drawn a b c will be the required angle. Although the angle a b c is not actually constructed when the lines are not drawn, nevertheless with the meaning we have agreed to give to geometry of the compass we shall call the angle constructed when the three points are given which determine it. Also when we say construct a polygon we understand-find all the points which suffice to determine the length and position of all the straight lines that it is necessary to draw to construct the complete polygon. The method depends on two fundamental and some preliminary construction problems which though simple are rather long because of the number of circles required to be drawn. The fundamental problems are as follows:

(1) To construct the point of intersection of two straight lines.

(2) To construct the point of intersection of a straight line and a circle

We shall now give the constructions and proofs for these problems and all other necessary preliminary ones.

Problem 1.: To get the inverse of a point P by compass alone.

Case 1. When $OP > \frac{r}{2}$

Solution:- (See figures XVl. and XVll)

Draw circle P (P O) and get points A and B.

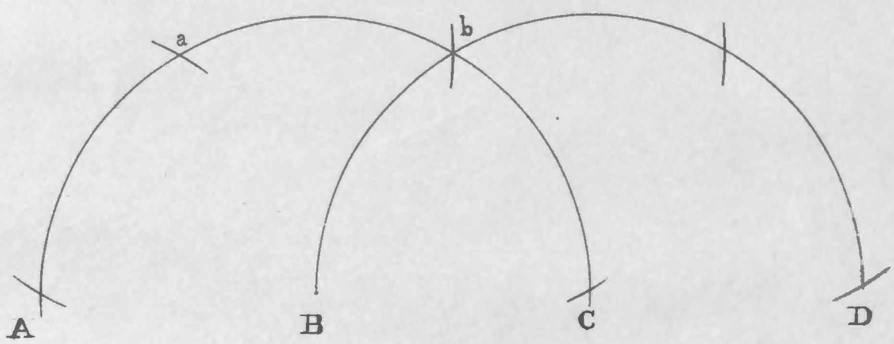


FIGURE XVIII

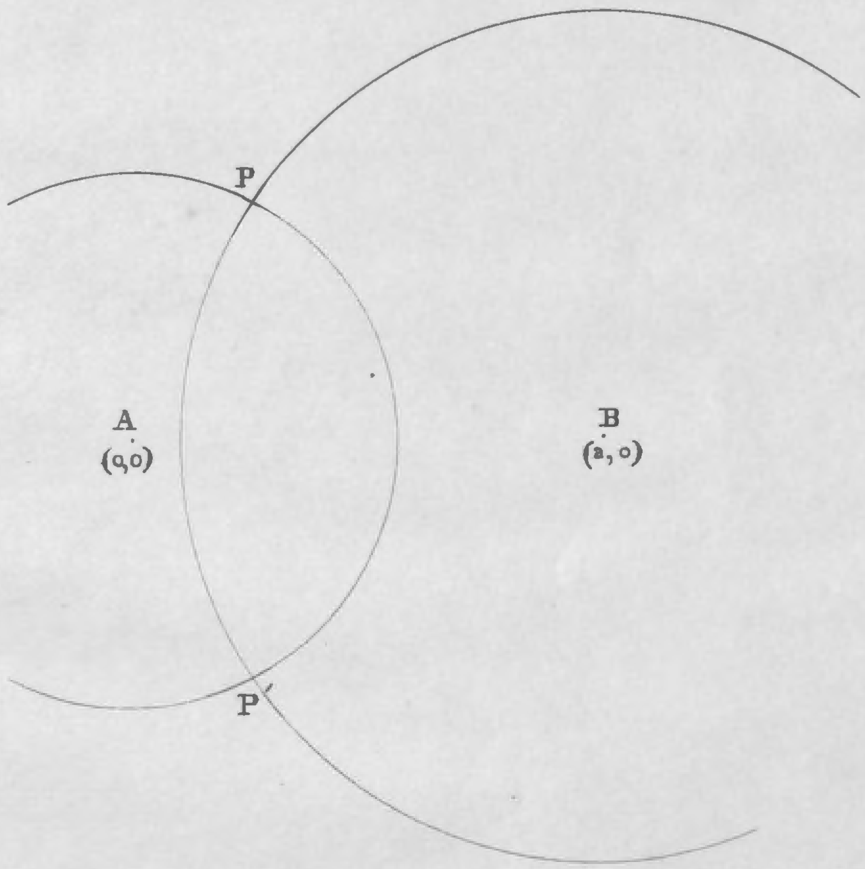


FIGURE XIX

Draw circles A (r) and B (r) which meet in Q.

Then Q is the inverse of P.

Proof:-

Let O P be x axis; $P = (a, 0)$

Then equation of circle P (O P) is $(x-a)^2+y^2=a^2$ (1)

and equation of circle of inversion O (r) is $x^2+y^2=r^2$ (2)

Solve (1) and (2) for x and y.

$$x = \frac{r^2}{2a}$$

$$y = \pm \frac{r}{2a}\sqrt{4a^2-r^2}$$

Then $A = \left(\frac{r^2}{2a}, \frac{r}{2a}\sqrt{4a^2-r^2}\right)$

$B = \left(\frac{r^2}{2a}, -\frac{r}{2a}\sqrt{4a^2-r^2}\right)$

Equation of circle A (r) is $\left(x-\frac{r^2}{2a}\right)^2+\left(y-\frac{r}{2a}\sqrt{4a^2-r^2}\right)^2=r^2$ (3)

Equation of circle B (r) is $\left(x-\frac{r^2}{2a}\right)^2+\left(y+\frac{r}{2a}\sqrt{4a^2-r^2}\right)^2=r^2$ (4)

Solve (3) and (4) for x and y, the coordinates of Q.

$$x = \frac{r^2}{a}; \quad y = 0$$

Then $O P \cdot O Q = a\frac{r^2}{a} = r^2$ Q.E.D.

Case 11. When $O P \leq \frac{r}{2}$

Cannot be proved until two lemmas have been proved.

Lemma 1.: To multiply a line segment A B by any integer n with compass alone. Solution:- (See figure XV111.)

Draw circle B (A B) and make A B = Aa = ab = b C.

Then A C = 2(A B)

In the same way double B C and get D so that A D = 3 (A B)

Continue in same way for mutiplying by any integer. Lemma 11. To construct with conpass alone the line from P perpendicular to A B i.e. find

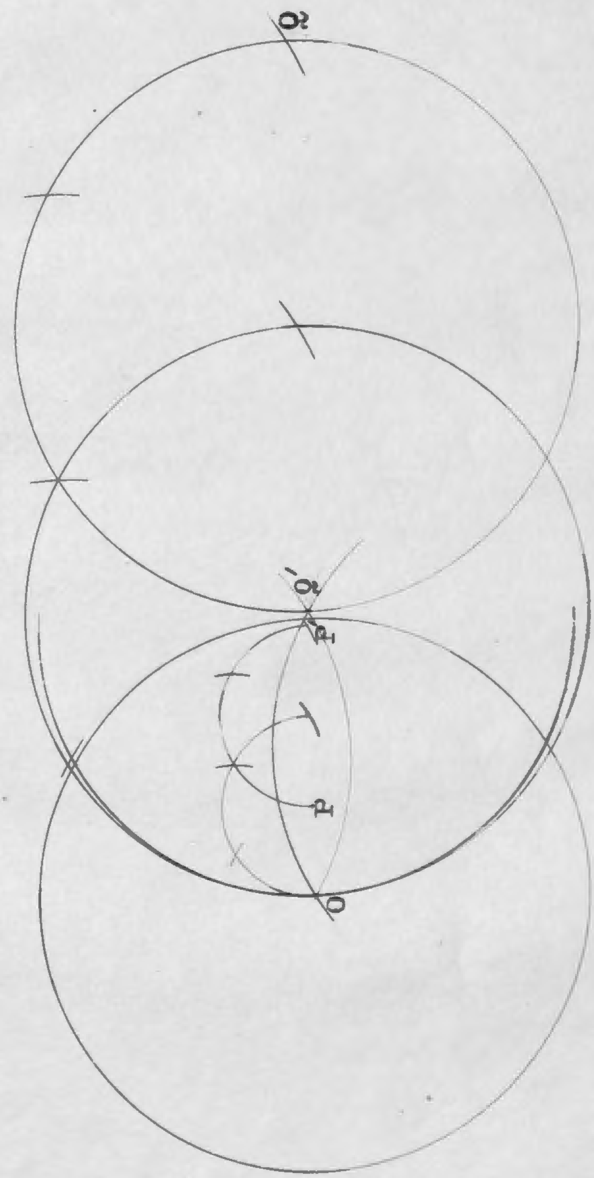


FIGURE XX

another point on the line such as P'.

Solution:- (See figure XIX)

Draw circles B (B P) and A(A P) which meet in P'.

P' is the required point.

Proof:-

P P' is the radical axis of circles B(B P) and A(A P)

Equation of circle A(A P) is $x^2+y^2=r^2=x'^2+y'^2$

Equation of circle B(B P) is $x^2+y^2-2ax-(x'^2-2ax'+y'^2)=0$.

$\therefore$ equation of radical axis is $2ax-x'^2-y'^2+x'^2-2ax'+y'^2=0$.

Equation of line joining centres A and B is $a\,y=0$.

$\therefore$ P P' is perpendicular to line A B. Q.E.D.

Scholium: P' is reflection of P in line A B because a diameter perpendicular to a chord bisects the chord;

Case 11. of Problem 1.: When $0P < \frac{r}{2}$

Solution:- (See figure XX, integer used is 3)

Multiply 0 P by 2,3,4,---or n and get a point P' such that

$OP' = n \cdot OP > \frac{r}{2}$ i e. $0P' = 3\,0P$

Construct Q', the inverse of P'. i.e. $0P' \cdot 0Q' = r^2$

or $0Q' = \frac{r^2}{0P'} = \frac{r^2}{3\,0P}$

or $3\,0Q' = \frac{r^2}{0P}$

Find Q such that

$0Q = n \cdot 0Q' = \frac{r^2}{0P}$ i e. $0Q = 3\,0Q' = \frac{r^2}{0P}$

and $0Q \cdot 0P = r^2$

Note: This method may sometimes be used to advantage for accuracy when

0 P is just a little greater than $\frac{r}{2}$ Then take $0P' = 2\,0P$ and P' will

be not far from the circumference. The following table in which x' represents the inverse of x, shows that points near the center when inverted involve a great error while points near the circumference when inverted give very small errors.

Let $x' = \frac{1}{x}$

Differentiating with respect to t (time) to show how x' changes when x changes we have

$$\frac{dx'}{dt} = \frac{-1}{x^2}\frac{dx}{dt}$$

If x is thought of as changing uniformly i.e. $\frac{dx}{dt} = 1.$

then $\frac{dx'}{dt} = \frac{-1}{x^2}$ from which we get

x	dx'
.1	100
.2	25
.3	11 1/9
.4	6 1/4
x	dx'
.5	4
.6	2.6
.7	2.04
.8	1.56
x	dx'
.9	1.23
1.0	1.
1.1	.82
1.2	.69

By experiment it seems to be just as accurate to get the inverse of P directly as to use the doubling and then invert. However it appears from the table that P', the inverse of a point P=.6, will involve an error 2.6 times as great as the error in P, but using the doubling process the error in P' will be only .69 times as much as the error in P. This apparent contradiction is probably accounted for by the fact that an error may be introduced in doubling.

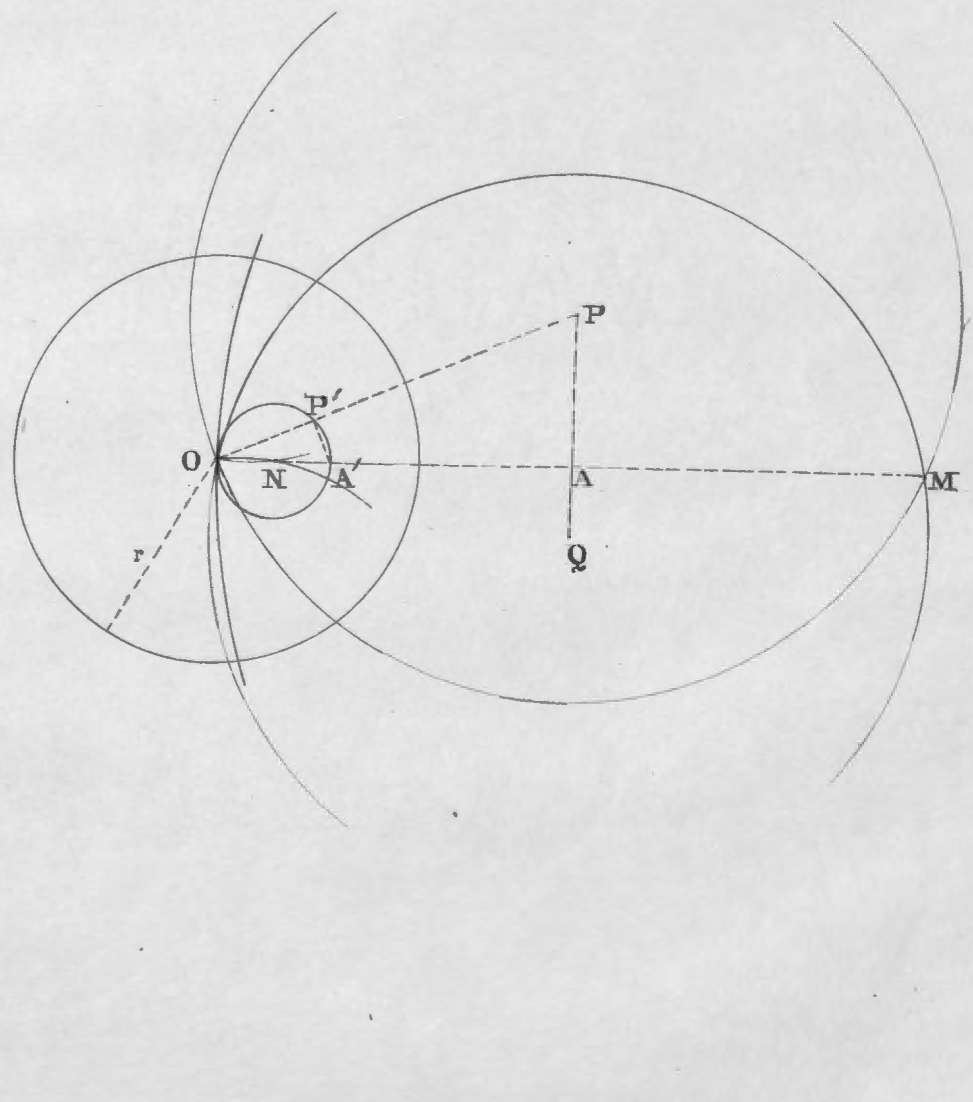


FIGURE XXI

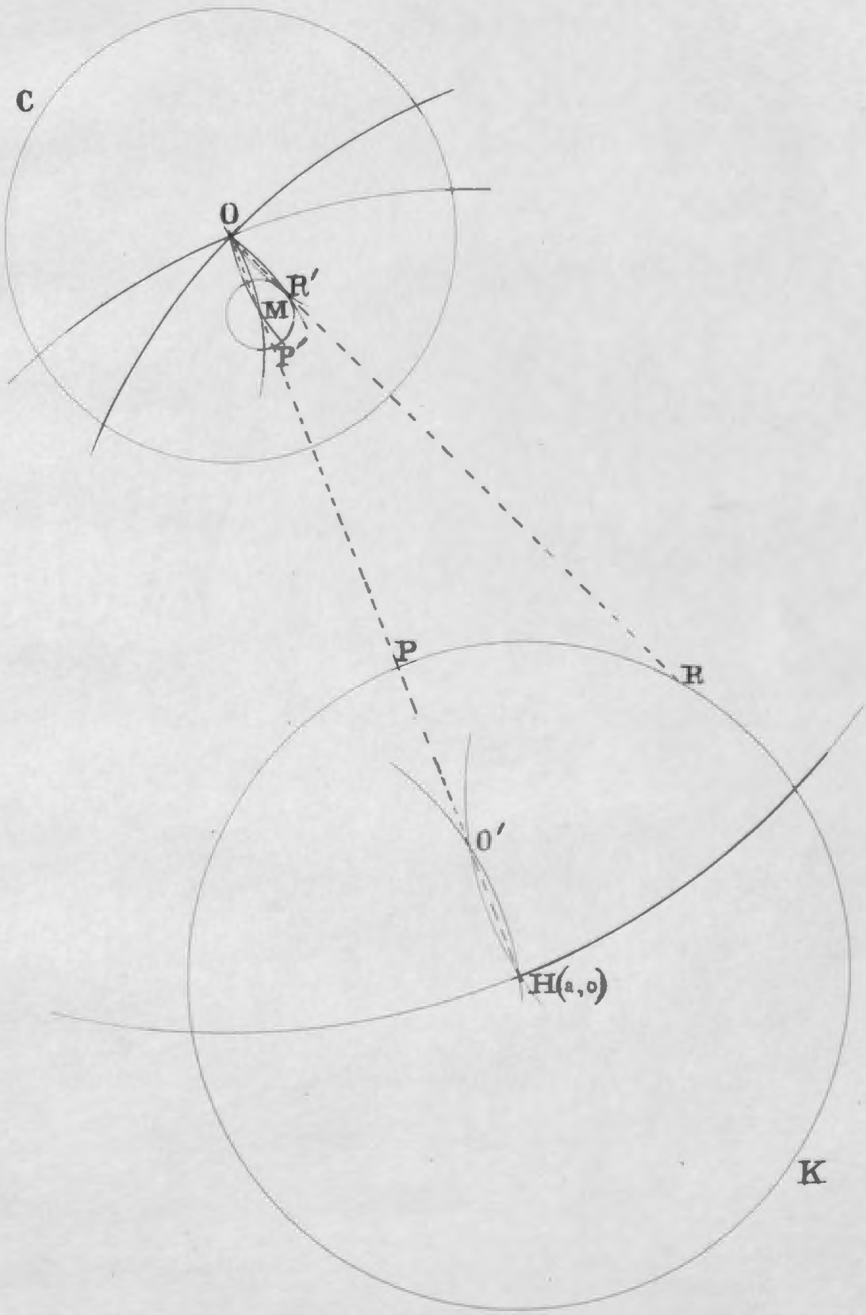


FIGURE XXII

Problem:11. – To construct with compass alone the circle which is the inverse of a given line determined by two points P and Q.

Solution:- (See figure XX1.)

Construct M the reflection of O in the given line P.Q.

Construct N the inverse of M.

Then N is the centre and N O the radius of the required circle.

Proof:-

By construction N is inverse of M and M is reflection of O.

$\therefore \quad OM \cdot ON = r^2$

and $\quad OM = 2OA$

but $\quad ON = \frac{1}{2} OA'$

$\therefore \quad OM \cdot ON = OA \cdot OA' = r^2$

In similar triangles O P' A' and O P A

$$\frac{OP'}{OA} = \frac{OA'}{OP}$$

$\therefore \quad OA \cdot OA' = OP \cdot OP'$

$\therefore \quad OM \cdot ON = OA \cdot OA' = OP \cdot OP' = r^2$ Q E.D.

We have already shown under the theory of inversion that the inverse curve is in general a circle.

Problem111.: To construct with compass alone the inverse of a circle K.

Solution:- (See figure XX11.)

Find O' the inverse of O with respect to K.

Find M the inverse of O' with respect to C.

Then M is center of required circle.

To construct the circle get one more point of it besides center

Take any point R of K.

Construct R' the inverse of R with respect to C.

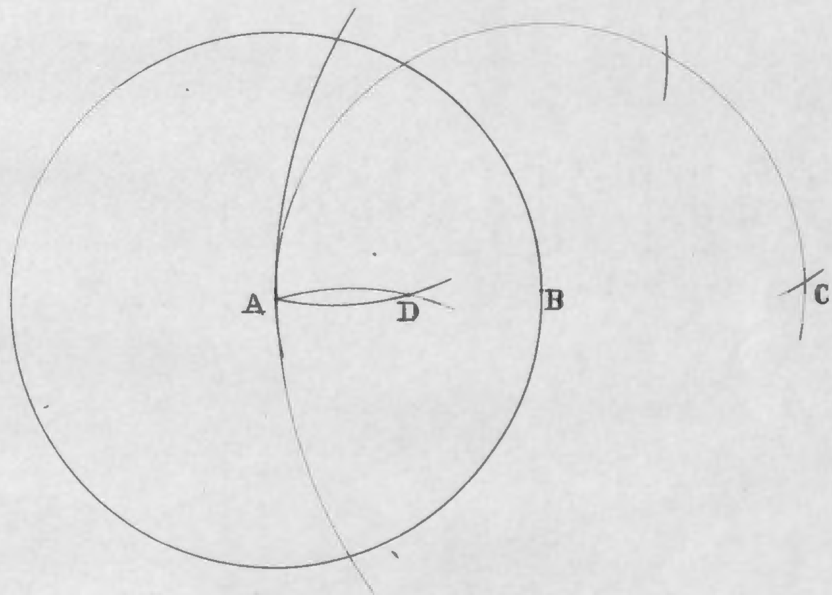


FIGURE XXIII

Then inverse of K is circle M (M R')

Proof:-

Let equation of K be $(x - a)^2 + y^2 = r^2$.

Substituting $x = \frac{x'}{x'^2 + y'^2}$ and $y = \frac{y'}{x'^2 + y'^2}$ we get the equation of the inverse curve is

$$\left(\frac{x}{x^2+y^2} - a\right)^2 + \left(\frac{y}{x^2+y^2}\right)^2 = r^2$$

Expanding and multiplying by x^2+y^2.

$$(a^2-r^2)\cdot x^2+(a^2-r^2)\cdot y^2 - 2ax+1=0.$$

or $\left(x - \frac{a}{a^2-r^2}\right)^2 + y^2 = \left(\frac{r}{a^2-r^2}\right)^2$ which is a circle

Center is $\frac{a}{a^2-r^2}$; radius is $\frac{r}{a^2-r^2}$

This proves that the inverse curve is a circle but we must also prove that it is this particular circle.

$$O H \cdot O' H = r^2$$

$$O' H = \frac{r^2}{a}$$

$$O O' = a - \frac{r^2}{a} = \frac{a^2 - r^2}{a}$$

$\therefore$ $O M = \frac{a}{a^2-r^2}$ which checks for the center.

By construction the inverse of P is P'

$$O P = a - r$$

$$\therefore \quad O P' = \frac{1}{a-r}$$

Then radius of inverse circle is $\frac{1}{a-r} - \frac{a}{a^2-r^2} = \frac{r}{a^2-r^2}$

which checks for radius.

<u>Problem lV.</u> To divide a line segment into equal parts by compass alone.

Case 1.: To divide into 2 equal parts.

Solution:- (See figure XXlll.)

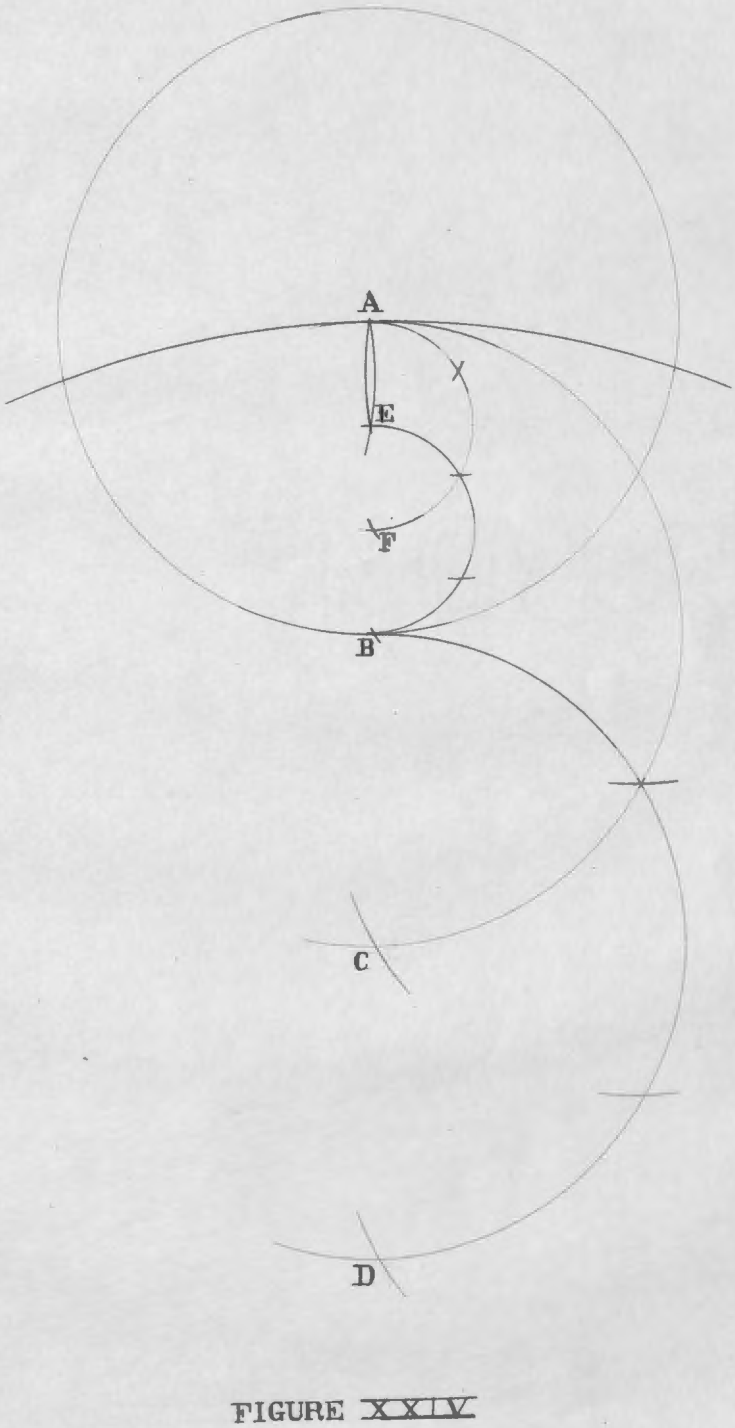


FIGURE XXIV

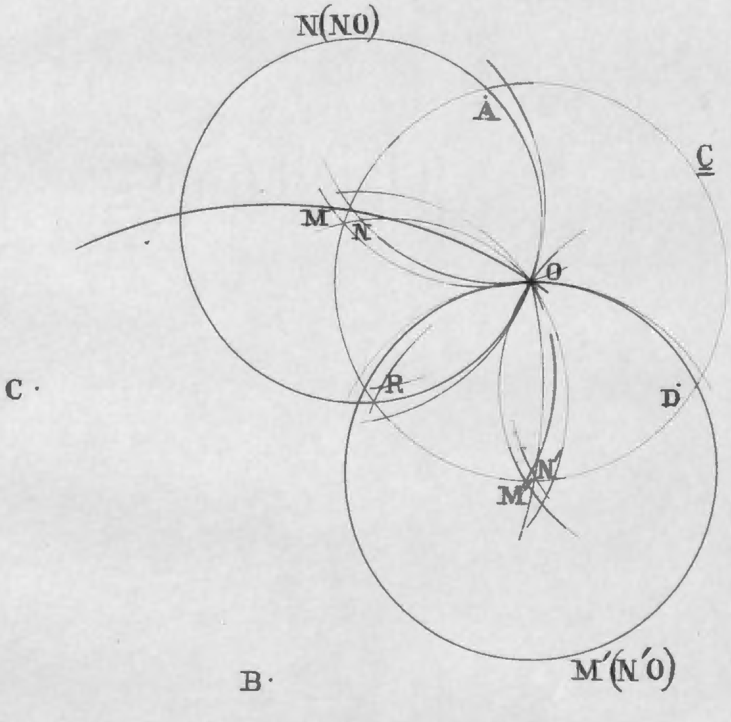


FIGURE XXV

Find point C suchthat $AC = 2AB$.

Find D the inverse of C with respect to A (A B)

Then D is required middle point.

Proof:-

$$AB \cdot AB = r^2 \qquad (1)$$

$$AC \cdot AD = r^2 \qquad (2)$$

but $$AC = 2AB$$

then $$2AB \cdot AD = r^2 \qquad (3)$$

Dividing (3) by (1) $AD = \frac{AB}{2}$ Q.E.D.

Case 11. To divide into 3 equal parts.

Solution:- (See figure XX1V.)

Multiply line A B by 3 and get point D.

Find E the inverse of D with respect to A (A B)

Double A E to A F.

Then E and F are the points which divide A B into thirds.

Proof is done in same way as Case 1.

Problem V. To construct with compass alone the point of intersection R of two lines each of which is given by two points.

Solution:- (See figure XXV.)

For position of C, the circle of inversion, imagine the bisector of the angle made by lines A B and D C.

Take center of C on bisector and radius of C a little less than twice the distance to A B.

Invert line A B into circle N (O N)

Invert line C D into circle N'(O N')

Invert point of intersection of N (O N) and N' (O N')

This will be required point of intersection R of lines A B and C D.

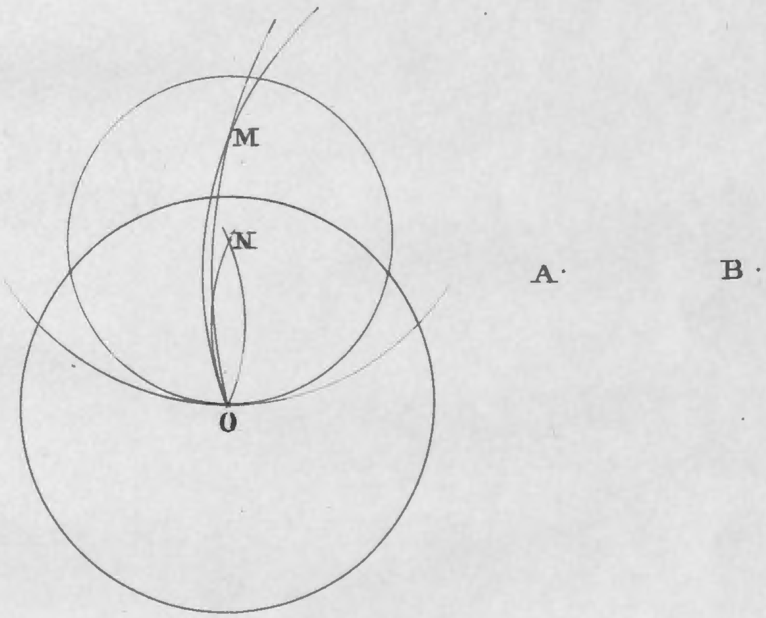


FIGURE XXVI

Problem Vl. : To construct with compass alone the point of intersection of a straight line and a circle when the straight line is given by two points and circle is given by center and radius i.e. the circle is completely drawn or easily can be.

Solution:- (See figure XXVl.)

Use the given circle as circle of inversion.

Invert line A B into circle N (N O)

The points where this circle N (N O) cuts the given circle are the required points.

Note:-

If line A B should go through the center of the given circle the above construction does not hold. In this case get the circle of inversion C as follows:

Let R, radius of circle of inversion, be a little less than $\frac{1}{2}$ A B.

O, centre of circle of inversion, be a little greater than $\frac{R}{2}$ from A B or A B produced and also outside the circumference of given circle K so that circumference of K cuts R a little more than $\frac{r}{4}$ from O where r is radius of circle K to be inverted.

Mascheroni's construction for this problem is much better than this.

Problem Vll. To add a given segment C D to a given line A B.

Solution:-
Double A B i e. find point C.

With B as center describe a circle of radius C D.

Find by problem Vl. the point where line B C cuts this circle. This will be the required point.

In order to show that there is no problem of elementary geometry which can not be solved by compass alone we need only observe that elementary geometry furnishes the means of finding the points of a problem either

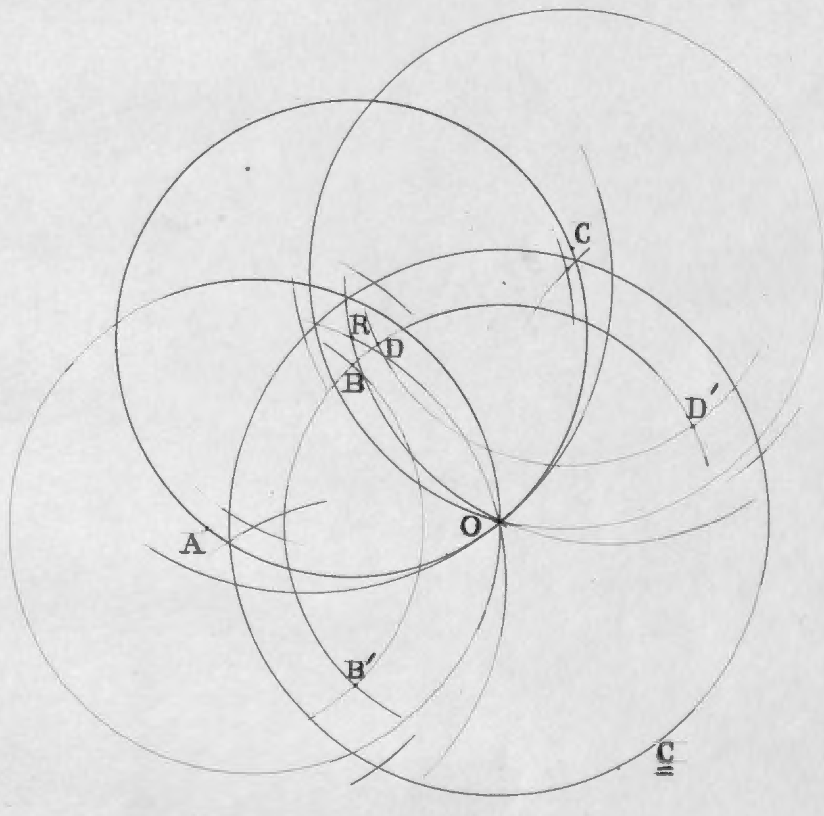


FIGURE XXVII

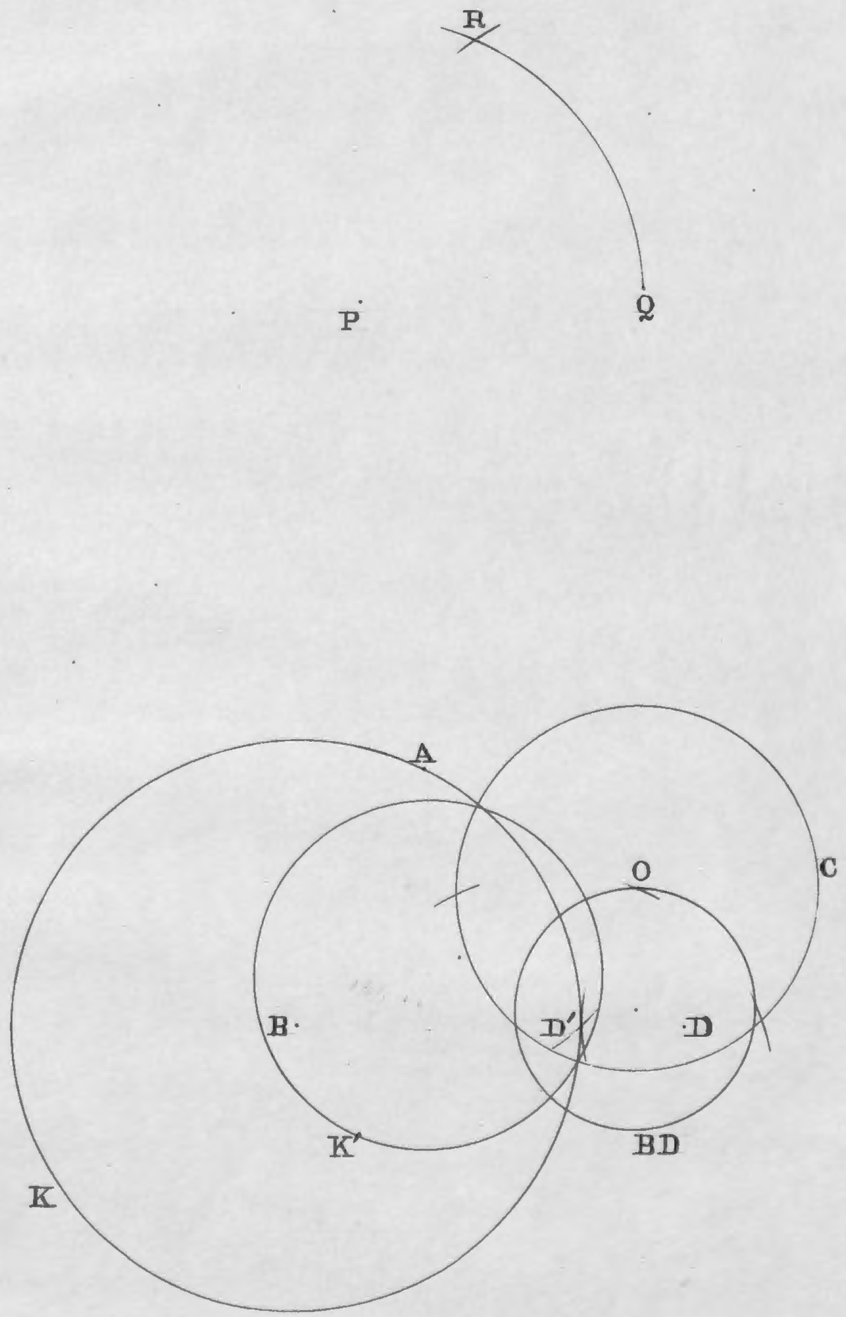


FIGURE XXVIII

by the intersection of arcs(and this is entirely permissable for geometry of the compass) or by the intersection of arcs and straight lines or by the intersection of straight lines. The last two cases are provided for in the geometry of the compass based on the theory of inversion in the two construction problems V. and Vl. just given.

We shall now give a few simple examples to show the use of the method.

Example l. (See figure XXVll.)

Given the circumference of a circle K; find its center R.

In Euclidean geometry i.e. using both ruler and compass this problem is solved by finding the point of intersection of the perpendicular bisectors of two sides of an inscribed triangle.

We shall therefore select three points A,O,C, on the circumference which determine the vertices of a triangle.

Find the perpendicular bisectors of A O and C O i.e. B B' and D D'.

Find R the point of intersection of B B' and D D' as in problem V.

Note:- To simplify the construction take the center of the circle of inversion at O, one of the points on the given circumference.

Example ll. (See figure XXVlll)

To construct an angle R P Q equal to a given angle A B D.

Draw circle B (B A).

Find D' where it cuts B D. (See note to problem Vl. for size and position of circle of inversion)

With P, vertex of new angle, as center and A B as radius describe a circle; with any point Q on this circumference as centre and radius A D' describe an arc cutting circumference in R.

Q P R will be required angle.

Note:-

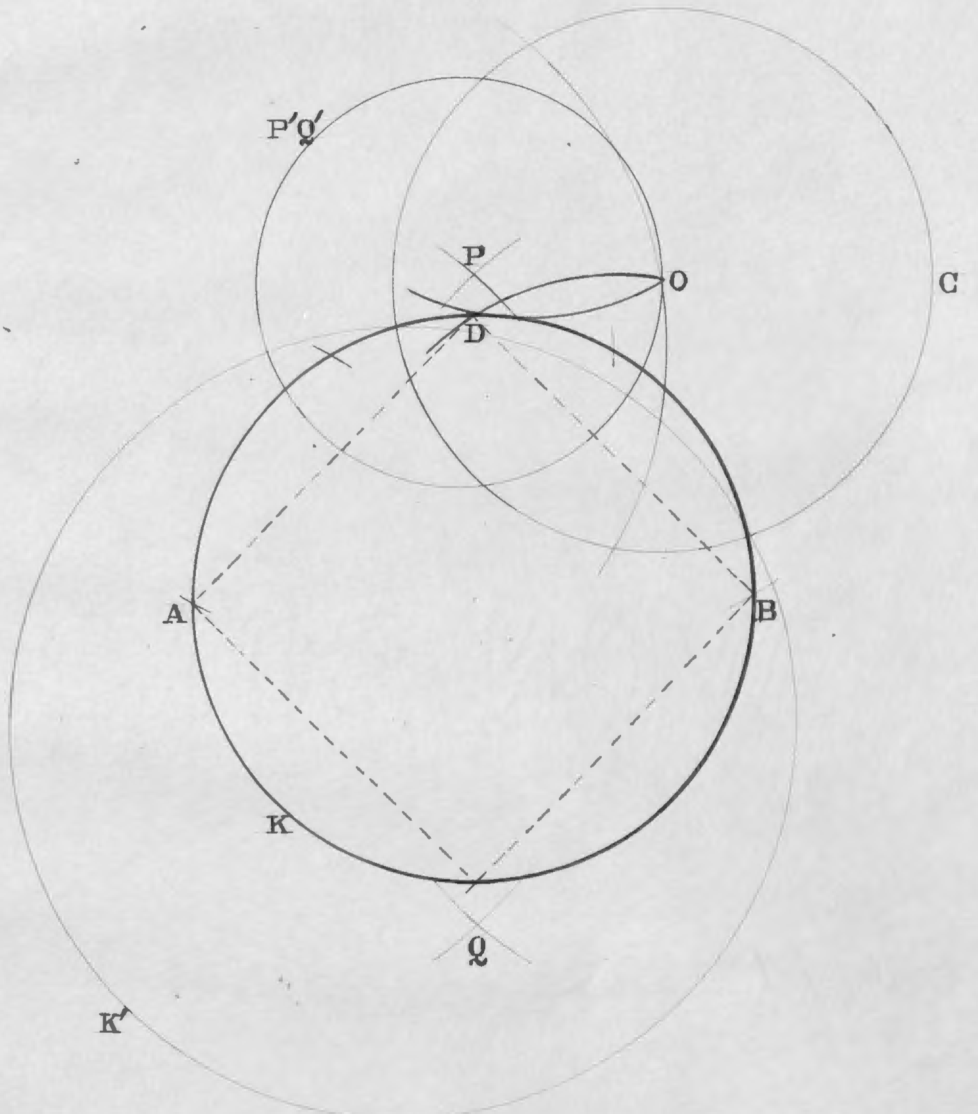


FIGURE XXIX

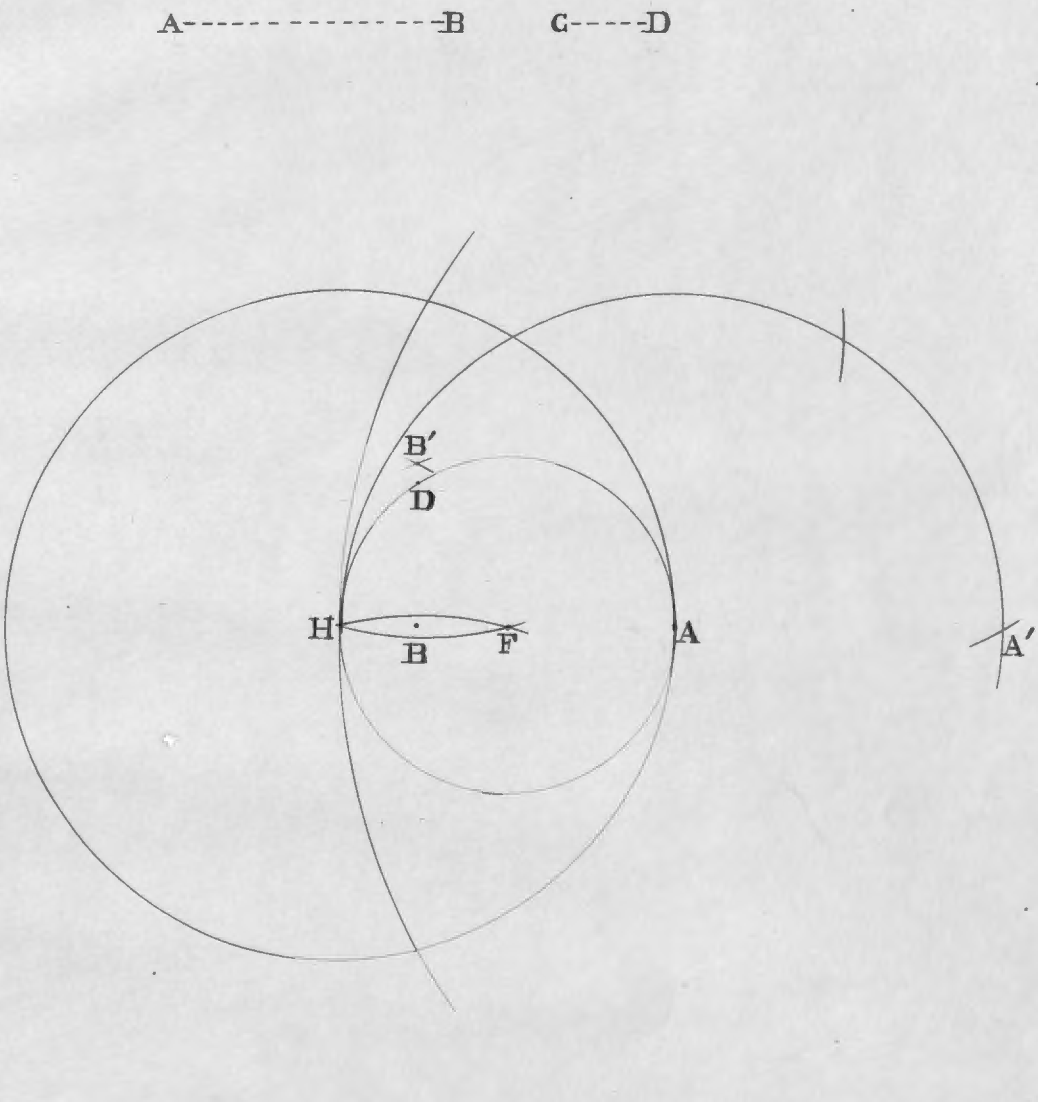


FIGURE XXX

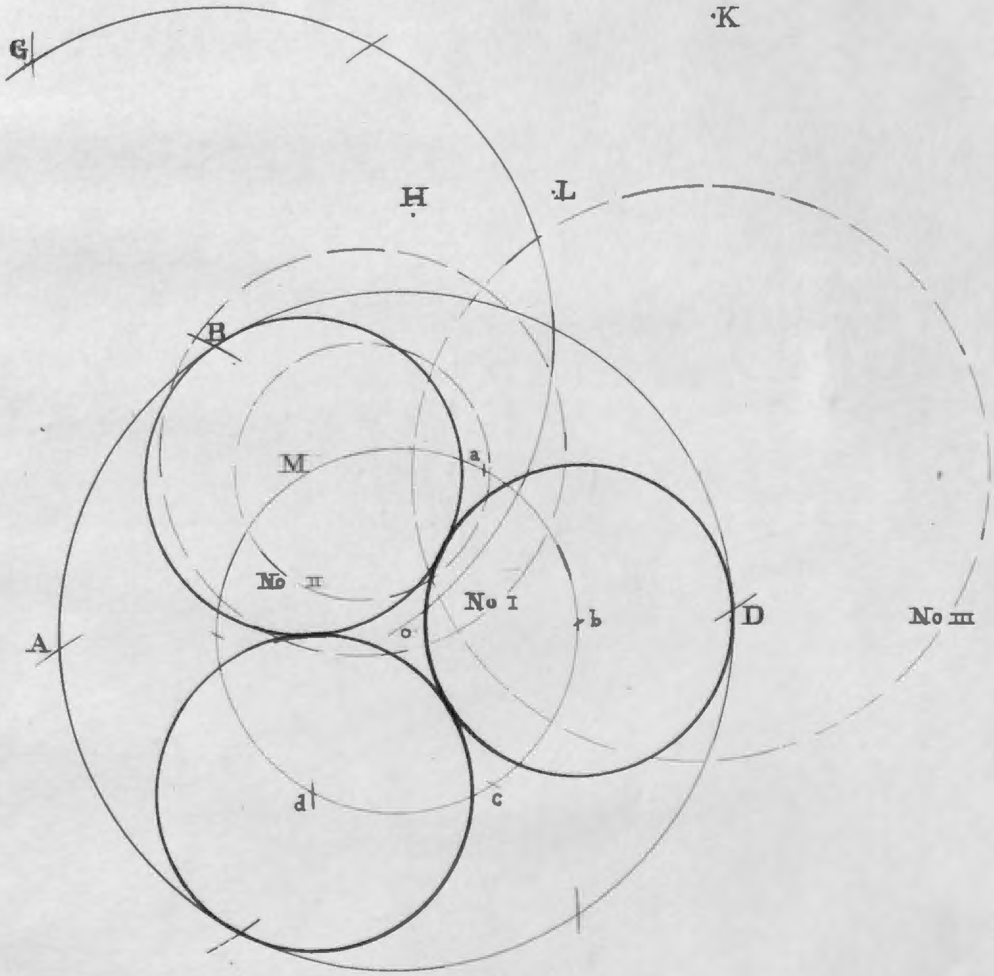


FIGURE XXXI

If one side P Q of required angle is given and is $\neq$ A B we must first find where P Q cuts circle P (A B) and then proceed as above.

Example 111. (See figure XX1X)

To inscribe a square in a circle.

Erect P Q perpendicular bisector of A B.

Find D where P Q cuts circle. (See note to problem V1. for size of C. of inversion)

A D = side of inscribed square.

Example 1V. (See figure XXX.)

Find the mean proportional between two given distances A B and C D.

Add the distance C D to A B and call the line A H. (Use problem V11)

Bisect A H at F (Use problem 1V.)

Draw circle F (F A)

Find B' another point on perpendicular to A H at B.

Find D, the point of intersection of F (F A) and B B'.

(F (F A) may be used as circle of inversion)

B D is required mean proportional.

Example V. (See figure XXX1.)

In a given circle whose center is O' to inscribe 3 equal circles tangent to each other and to the given circle.

Divide the circumference into six equal parts at points A, B, C, D, E. F.

One of the required circles is that inscribed in the triangle formed by the tangent to the circle at B and the radii O' A, O' C produced.

The center of this circle is the point of intersection of the bisectors of two of the angles while the radius is the distance from B to this point of intersection.

Construction to get the point of intersection of the bisectors of the angles requires three different circles of inversion.

Find another point on bisector of angle A O' C i.e find another point on radius O'B.

Erect the perpendicular to O' G at B i.e. find point H.

Find K the point of intersection of B H and O' C.

Bisect angle B K O' i.e. find a point L on the bisector.

Find M the intersection of K L and B O'.

Then M (M B) is one of required circles.

For second inscribed circle draw circle O' (O' M) and make O' M = M a =

a b = b c = c d.

d (M B) is second inscribed circle.

b (M B) is third inscribed circle.

Euclid has solved a great many problems using the compass alone. I shall not here consider any of those problems as they do not depend on the theory of inversion. For example to draw through a given point a line parallel to a given line, solved in the same way by Euclid and Mascheroni (See figure Vl.) is such a problem

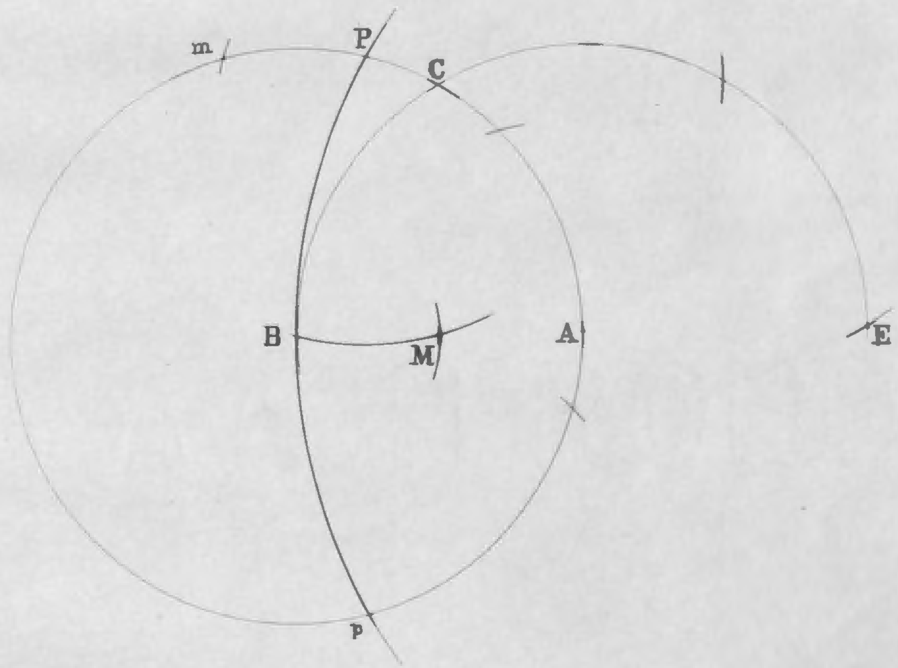


FIGURE XXXII

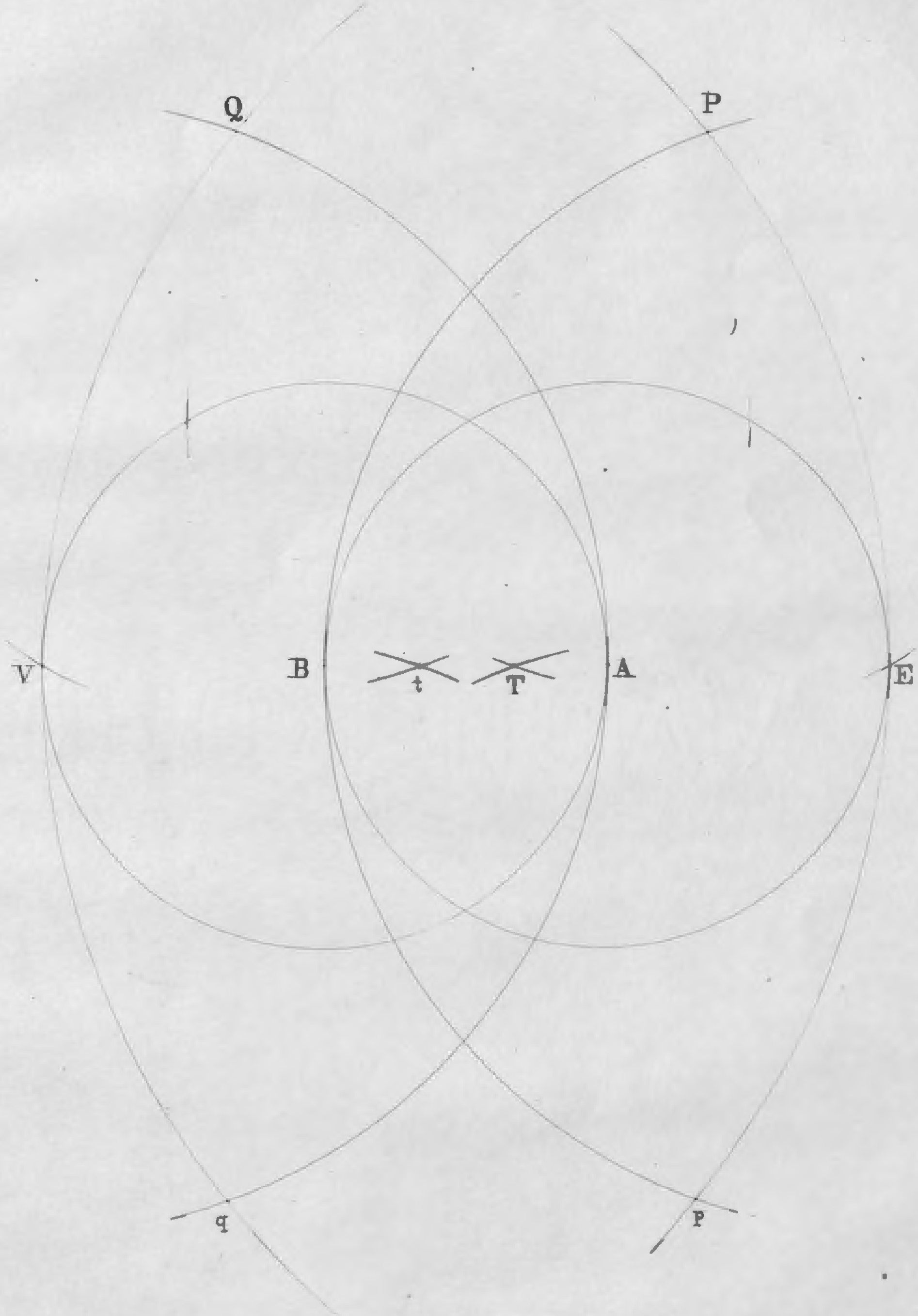


FIGURE XXXIII

Comparison of Mascheroni's and Inversion Constructions.

For dividing a given line A B into two equal parts Mascheroni gives five different solutions, one of which is the inversion construction. For comparison we shall give here his first solution which differs from the inversion construction in the last step only. The method based on the theory of inversion (See Problem lV.) is somewhat simpler requiring the drawing of four less arcs but Mascheroni's method is a little more exact in practice, because the circles cut at angles closely approaching right angles while in the other method they sometimes cut at very small angles.

Mascheroni's first solution for dividing the line A B into two equal parts (See Figure XXXll.)

Describe the semicircumference B C D E.

Describe arc p B P of circle E (E B)

Describe semicircumference p A P m of circle B (B A)

Describe arc B M of circle P (P B)

Make P m=B M.

Then point M is required point.

Very closely allied to this method of bisecting a line is the construction for dividing a line into three equal parts. By examining Mascheroni's figure (See figure XXXlll) it is easily seen that his solution, which requires that sixteen circles be drawn, can be simplified by applying inversion for using E (E B) as circle of inversion, T is really the inverse of V.

Mascheroni's construction:-

Double A B to V and E.

Describe arc q V Q of circle E (E V) and arc p E P of circle V (E V). Cut these arcs in q,Q,p,P by arcs of circles V (E B) and E (E B)

Describe arcs p (E B), P (E B) which cut in T, also arcs q (E B), Q (E B) which cut in t.

T and t are required points.

The proof which we shall not give depends on the similarity of triangles. The inversion solution of this problem, besides having the advantage of simplicity, serves as a general theorem, i.e. the method is one which can be extended to the problem of dividing into any number of equal parts. The inversion solution is as follows:

Multiply the line by the integer which represents the desired number of equal parts and find the inverse of the extremity of the line with reference to A (A E).

Likewise Mascheroni's method for dividing a line into any number of equal parts can be simplified by inversion, for not only the construction but also the proof involves the theory of inversion. The only advantage of Mascheroni's method is that it is sometimes a little more accurate. Note:- For either method the construction is simplified considerably if the straight line to be divided is actually drawn instead of being only determined by points.

Suppose we wished to erect the perpendicular to A B at B i.e. to erect the perpendicular to a line at one extremity of it. The simplest direct way by Compass alone to solve the problem would be to double the line and erect the perpendicular at its middle point as in Euclidean geometry. If for any reason we could not extend the line to the desired length, we should have to invert the construction of Euclidean geometry by which this problem is solved, which would be a rather long and usually inaccurate construction. In this case however Mascheroni's solution would be super-

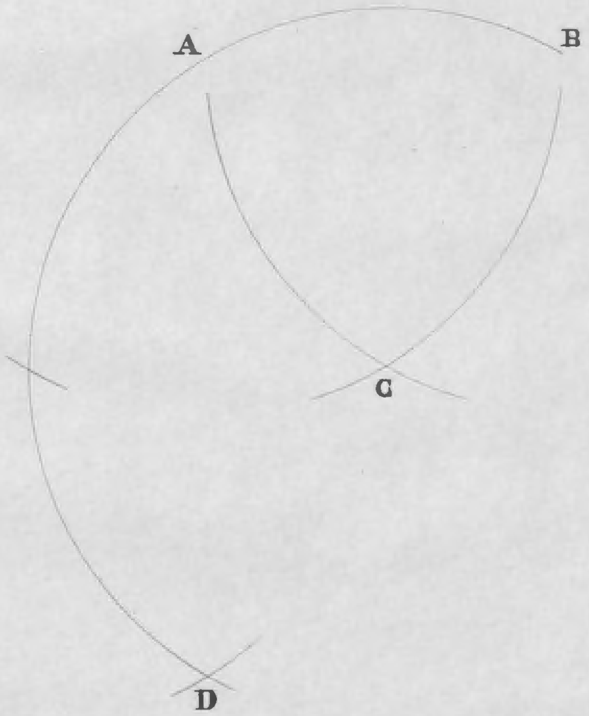


FIGURE XXXIV

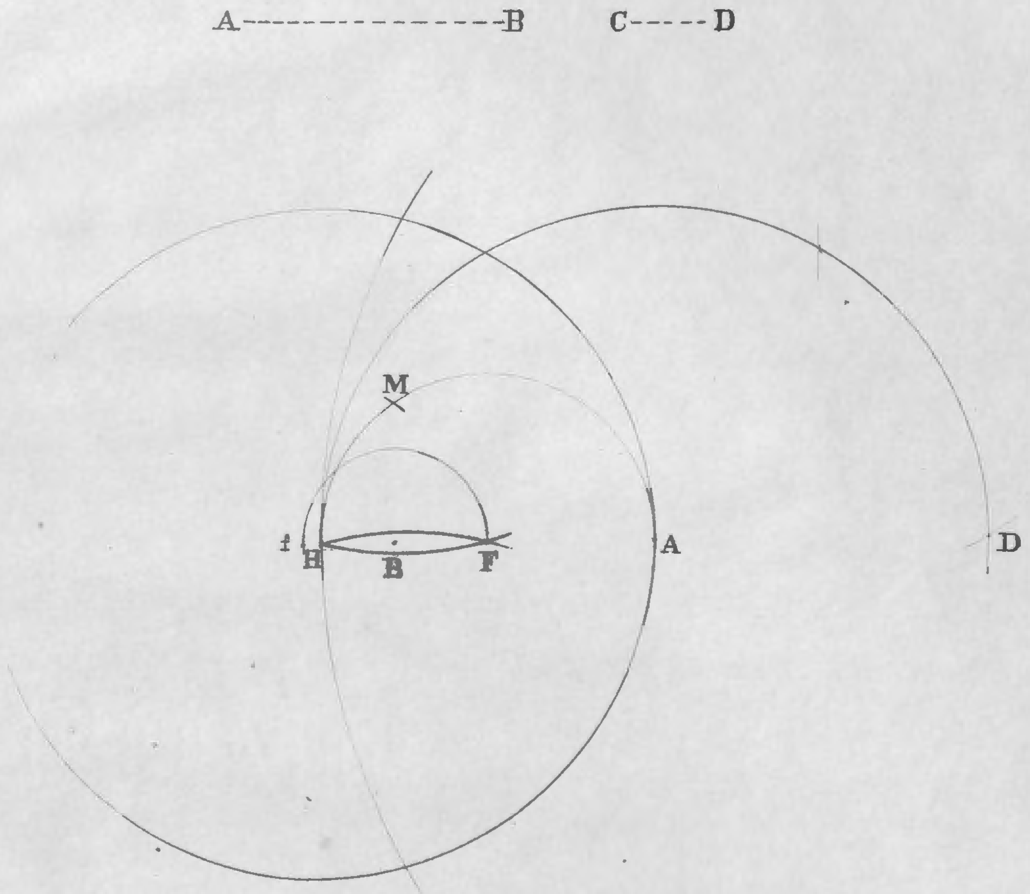


FIGURE XXXV

ior both for simplicity and exactness.

Mascheroni's solution to erect the perpendicular to A B at A, i.e. find point D so that D A is perpendicular to A B at A is as follows: (See figure XXXIV.)

Describe circles A (A B) and B (A B) intersecting at C.

Describe circle C (A B) and determine semicircumference B A D.

D is the required point.

Example IV. Page 58, to find the mean proportional between the two given distances, could have been simplified considerably by using Mascheroni's construction which avoids the long process of finding the intersection of a line and a circle, since he uses a radius which gives him as a point on the perpendicular, a point which is also on the circumference of the circle and therefore the required point. (See figure XXXV)

A H is the sum of A B and C D.

F is the middle point of A H.

B F = B f.

Circles f (A F) and F (A F) cut at M.

B M is the required mean propertional.

In order to inscribe a regular decagon in a circle, with ruler and compass we must know how to divide a line internally into extreme and mean ratio. No one who compares the following solution, Mascheroni's construction, with that given in any ordinary Euclidean Geometry will hesitate to say that the problem is more easily solved by compass alone than by the ordinary ruler and compass construction. However, the proof is exceedingly complicated and will not be given here.

Mascheroni's Solution for dividing a line A B internally into extreme and

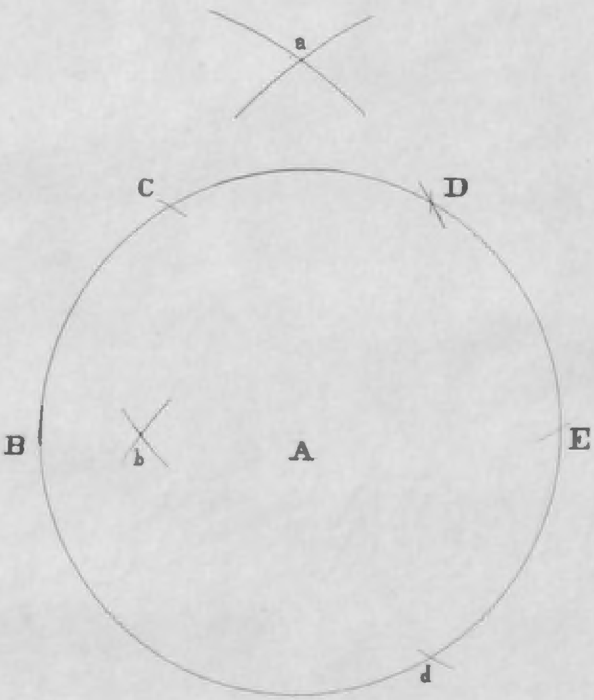


FIGURE XXXVI

mean ratio. (See figure XXXVI.)

Describe circle A (A B),

Make A B = B C = C D = D E = E d.

Also B D = B a = E a

and A a = D b = d b.

Then A B will be divided into extreme and mean ratio at b.

The two fundamental problems V and VI.(Page 49) which together with the theory of inverting any line into a circle and any circle into another circle give us a general method of solving any problem by compass alone may also be considered as Mascheroni's fundamental constructions. If the line of problem VI. does not go through the center of the given circle Mascheroni's construction (see Page 21) is not very much simpler than that given; however if the line does go through the center Mascheroni's construction (see Page 21) is for superior both for accuracy and simplicity since the given circle cannot be used as circle of inversion. The only objection to adopting his method of finding the intersection of a circle and a line through the center is that one must also adopt his method of bisecting an arc which in turn depends on other constructions which have not been studied.
The solution given for the other fundamental problem, i.e. problem V. requires the drawing of sixteen arcs while Mascheroni's solution of the same problem (See Page 21) requires only eleven. However, the method based on inversion is superior to Mascheroni's first because his depends on so many problems, which are solved not by any theory but by chance, and second because the solution given being based on the theory of inversion requires no further proof while his proof is long and complicated.

Were it not for the fact that some constructions necessarily involve the intersection of arcs at very small angles geometry of the compass would be far superior in point of accuracy to that of ordinary geometry To show in general the superiority of the usage of the compass over that of the ruler when it is a question of precision one need only observe that with a ruler, no matter how short, it is almost impossible to guarantee the precision of all points in the line drawn. Even though the ruler were absolutely straight it is well known that the trace of a line drawn the length of a ruler carries with it an uncertainty of parallelism in the movement of the axis of the marking point or of perfect applicationof this point to the edge of the ruler. The Compass is not subject to these inconveniences; it is only necessary that the compass be set and the two points of it be very fine. Placing one of them on a point taken as center, the other describes an arc which is as exact as possible. Let us take for example the problem to divide an inch into five equal parts. With the ordinary ruler alone this is impossible, by the method used in Euclidean geometry using ruler and compass, viz.divide any convenient length into five equal parts and then lay off the proportional parts on the given line the errors of non-parallelism and movement of the marking point in tracing straight lines make the construction far less accurate than that made by compass alone based on the theory of inversion. However in geometry of the compass if small errors are made at the beginning of a problem they are multiplied and carried through the rest of the construction making it far from accurate. Let us take for example the problem to find the intersection of two straight lines. By ruler alone this is easily accomplished without much error by a good straight edge. When we remember that to make this construction by compass alone it is necessary to draw sixteen circles it is evident that errors can very easily be made and many times multiplied.

In a great many, and I might safely say in the majority of instances the simplicity of Mascheroni's method over that based on inversion is plainly evident provided no account is taken of the time spent in first discovering the solution. Any problem, the solution of which by ruler and compass is known, has so apparent a compass construction based on the theory of inversion that it is practically solved; but not so with Mascheroni's method which depends largely on chance for a solution.